Math 2280 - Lecture 39

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In today's lecture, we'll investigate how Fourier series can be used to solve differential equations of the form:

$$mx'' + kx = F(t),$$

and the properties of these solutions.

Today's lecture corresponds with section 9.4 of the textbook. The assigned problems are:

Section 9.4 - 1, 2, 3, 19, 20

Applications of Fourier Series

Let's investigate the situation of undamped motion of a mass m on a spring with Hooke's constant k under the influence of a *periodic* force $F(t)$. As we've learned, the displacement from equilibrium satisfies:

$$mx''(t) + kx(t) = F(t).$$

The general solution to this system will be an equation of the form:

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + x_p(t),$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the system and $x_p(t)$ is a particular solution to the differential equation. Here we want to use Fourier series to find a *periodic* particular solutions of the differential equation, which we will denote $x_{sp}(t)$ and call the *steady periodic solution*.

We will assume for simplicity that $F(t)$ is an odd functions with period $2L$, so its Fourier series has the form

$$F(t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi t}{L}.$$

If $n\pi/L$ is not equal to ω_0 for any positive integer n, we can determine a steady periodic solution of the form

$$x_{sp}(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$$

by substituting the series into our differential equation and equating the coefficients. Let's see an example of how to do this.

Example - Suppose that $m = 2kg$, $k = 32N/m$, and that $F(t)$ is the odd periodic force with period $2s$ given in one period by

$$F(t) = \begin{cases} 10N & 0 < t < 1; \\ -10N & 1 < t < 2. \end{cases}$$

Solution - By essentially the same computation we did three lectures ago, we can calculate that the Fourier series of $F(t)$ is

$$F(t) = \frac{40}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left((2n+1)\pi t\right)}{2n+1}.$$

If we plug this into the right side of our differential equation, and plug in the solution

$$x_{sp}(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t,$$

we get

$$\sum_{n=1}^{\infty} b_n(-2n^2\pi^2 + 32) \sin n\pi t = \frac{40}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left((2n+1)\pi t\right)}{2n+1}.$$

Equating coefficients we get $b_n = 0$ for n even, and for n odd we get

$$b_{2n+1} = \frac{20}{(2n+1)\pi(16 - (2n+1)^2\pi^2)}.$$

So, our steady periodic solution is

$$x_{sp}(t) = \frac{20}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left((2n+1)\pi t\right)}{(2n+1)(16 - (2n+1)^2\pi^2)}.$$

Now, what happens if $n\pi/L$ happens to equal ω_0 for some value of n? In this case, we get resonance, or, more precisely, pure resonance. The reason is that the equation

$$mx'' + kx = B_N \sin \omega_0 t$$

has the resonance solution

$$x(t) = -\frac{B_N}{2m\omega_0} t \cos \omega_0 t$$

if $\omega_0 = \sqrt{k/m}$. The particular solution we get using Fourier series methods is then

$$x(t) = \frac{-B_N}{2m\omega_0} t \cos \omega_0 t + \sum_{n \neq N} \frac{B_N}{m(\omega_0^2 - n^2\pi^2/L^2)} \sin \frac{n\pi t}{L}.$$

Example - Suppose that $m = 2$ and $k = 32$. Determine whether pure resonance will occur if $F(t)$ is the odd periodic function defined in one period to be:

(a) - $F(t) = \begin{cases} 10 & 0 < t < \pi \\ -10 & \pi < t < 2\pi \end{cases}$

(b) - $F(t) = 10t$, for $-\pi < t < \pi$.

Solution -

(a) - The natural frequency is $\omega_0 = 4$, and the Fourier series of $F(t)$ is

$$F(t) = \frac{40}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \cdots \right).$$

Pure resonance does not occur because there is no $\sin 4t$ term in the Fourier series.

(b) In this case the Fourier series is

$$F(t) = 20 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt.$$

Pure resonance *does* occur because of the presence of the term containing the factor $\sin 4t$.

Even if we don't have resonance, we can have *near resonance*, where a single term in the solution has a frequency that is close to the natural resonant frequency, and is magnified.

Example - Find a steady periodic solution to

$$x'' + 10x = F(t),$$

where $F(t)$ is the period 4 function with $F(t) = 5t$ for $-2 < t < 2$ and Fourier series

$$F(t) = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi t}{2}.$$

Solution - Following the same procedure we used in the first example we obtain the relation

$$\sum_{n=1}^{\infty} b_n \left(-\frac{n^2\pi^2}{4} + 10 \right) \sin \frac{n\pi t}{2} = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi t}{2}.$$

We equate coefficients of like terms and then solve for b_n to get the steady periodic solution

$$x_{sp}(t) = \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(40 - n^2\pi^2)} \sin \frac{n\pi t}{2}$$

$$\approx (.8452) \sin \frac{\pi t}{2} - (24.4111) \sin \frac{2\pi t}{2} - (0.1738) \sin \frac{3\pi t}{2} + \cdots .$$

The very large magnitude of the second term is because $\omega_0 = \sqrt{10}$ is very close to $\pi = \frac{2\pi}{2}$.