Math 2280 - Lecture 36

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The last concept we'll talk about in this class is Fourier series, which are some of the most interesting and most useful objects (or methods, or whatever) in mathematics. Fourier series are used all the time for both practical and theoretical mathematics, and there are whole advanced[1] classes on the subject. So, needless to say, in the next two weeks we'll only be putting our toe in the ocean. However, we can learn enough to get some useful and interesting results, and to understand the basic idea behind the method.

Today's lecture corresponds with section 9.1 of the textbook. The assigned problems are:

Section 9.1 - 1, 8, 11, 13, 21

Periodic Functions and Trigonometric Series

Let's begin by taking a look at a relatively simple differential equation that we've met before:

$$x''(t) + \omega_0^2 x(t) = f(t).$$

We've learned how to solve this ODE for a number of possible functions $f(t)$. We know the solution for $f(t) = 0$, or when $f(t)$ is made up of sums and products of exponentials, polynomials, sines, and cosines. As a specific example, suppose:

[1] And very advanced. And very, very advanced.

$$f(t) = A\cos(\omega t).$$

We learned a while ago that a particular solution to this ODE is:

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t)$$

as long as $\omega_0 \neq \omega$. Using the linearity of our differential equation (a property which becomes incredibly important in the context of Fourier series) we can induce from this solution that if:

$$f(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t),\ \omega_n \neq \omega_0 \text{ for any } n.$$

then our particular solution will be:

$$x_p(t) = \sum_{n=1}^{N} \frac{A_n}{\omega_0^2 - \omega_n^2}\cos(\omega_n t).$$

Nothing new here, but this observation is the starting point for the study of Fourier series. What is says is that for any function $f(t)$, if it can be represented as a sum of cosine functions, then we know how to solve it. This idea can be immediately extended to functions that can be represented as sums of sine and cosine functions. But, how many functions can be represented as sums of sine and cosine functions? Well, if you allow infinite sums, quite a few!

Periodic Functions

A function is periodic with period p if there exists a number $p > 0$ such that:

$$f(t+p) = f(t) \text{ for all } t.$$

The smallest such p, if a smallest one exists, is called *the* period (also sometime the *fundamental period*) of the function.

For any piecewise continuous function $f(t)$ of period 2π we can define its *Fourier series*:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos{(nt)} + b_n \sin{(nt)}).$$[2]

The coefficients of this series are defined by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos{(nt)}dt$$

for $n = 0, 1, 2, \ldots$, and:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin{(nt)}dt$$

for $n = 1, 2, \ldots$.

Calculating Fourier Transforms

Let's first get down a few facts about integrals of this kind.

$$\int_{-\pi}^{\pi} \cos{(mt)}\cos{(nt)}dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases};$$

$$\int_{-\pi}^{\pi} \sin{(mt)}\sin{(nt)}dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases};$$

$$\int_{-\pi}^{\pi} \cos{(mt)}\sin{(nt)}dt = 0 \text{ always.}$$

[2]Note that right now this is just a definition. We're not making any claims yet about how it can be used to represent $f(t)$. That's coming later.

Proving these just involves some clever use of trigonometric identities. As an example, here's how you'd prove the second relation.

Proof - We first note the trigonometric identities:

$$\cos((m+n)t) = \cos(mt)\cos(nt) - \sin(mt)\sin(nt)$$

and

$$\cos((m-n)t) = \cos(mt)\cos(nt) + \sin(mt)\sin(nt).$$

Using these relations we get:

$$\sin(nt)\sin(mt) = \frac{\cos((m-n)t) - \cos((m+n)t)}{2}.$$

Therefore, our integral becomes:

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \int_{-\pi}^{\pi} \frac{\cos((m-n)t) - \cos((m+n)t)}{2} dt.$$

If $m \neq n$ this integral evaluates to:

$$\frac{1}{2}\left[\frac{\sin((m-n)t)}{(m-n)} - \frac{\sin((m+n)t)}{(m+n)}\right]\Bigg|_{-\pi}^{\pi}$$
$$= \frac{1}{2}(0-0) - \frac{1}{2}(0-0) = 0.$$

On the other hand, if $m = n$, then our integral is:

$$\int_{-\pi}^{\pi} \sin^2 nt dt = \int_{-\pi}^{\pi} \frac{1-\cos 2nt}{2} dt = \left(\frac{t}{2} - \frac{\sin 2nt}{4n}\right)\Bigg|_{-\pi}^{\pi} = \pi.$$

Example - Find the Fourier series of the square-wave function:

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \\ 0 & t = \{-\pi, 0, \pi\} \end{cases}$$

Solution - The Fourier series is:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_{-\pi}^{0} dt + \frac{1}{\pi}\int_{0}^{\pi} dt \\ &= -\frac{1}{\pi}(\pi) + \frac{1}{\pi}(\pi) = -1 + 1 = 0. \end{aligned}$$

For $n > 0$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \frac{1}{\pi}\int_{-\pi}^{0} (-\cos(nt))dt + \frac{1}{\pi}\int_{0}^{\pi} \cos(nt)dt \\ &= -\frac{\sin(nt)}{n\pi}\bigg|_{-\pi}^{0} + \frac{\sin(nt)}{n\pi}\bigg|_{0}^{\pi} = 0. \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = -\frac{1}{\pi}\int_{-\pi}^{0} \sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} \sin(nt)dt \\ &= -\frac{1}{\pi}\left(\frac{-\cos(nt)}{n}\right)\bigg|_{-\pi}^{0} - \frac{1}{\pi}\left(\frac{\cos(nt)}{n}\right)\bigg|_{0}^{\pi} = \frac{2}{n\pi}(1 - \cos(n\pi)) \\ &= \frac{2}{n\pi}[1 - (-1)^n]. \end{aligned}$$

So,

$$b_n = \begin{cases} 0 & even \\ \frac{4}{n\pi} & odd \end{cases}$$

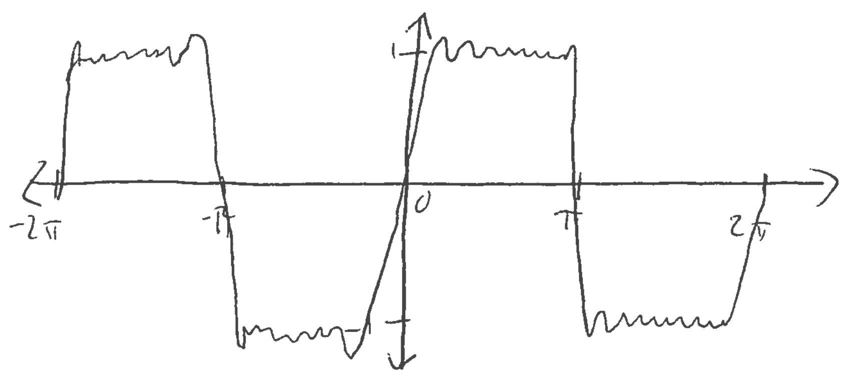
Therefore, the Fourier transform of the function $f(t)$ is:

$$f(t) \sim \frac{4}{\pi} \sum_{odd} \frac{\sin(nt)}{n} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)t}{n}.$$

The partial sums of this series are:

$$S_N(t) = \frac{4}{\pi} \sum_{n=0}^{N} \frac{\sin((2n+1)t)}{2n+1}.$$

The graph of one of these partial sums looks like:



Example
Find the Fourier transform of the 2π-periodic function:

$$f(t) = \begin{cases} 3 & -\pi < t \leq 0 \\ -2 & 0 < t \leq \pi \end{cases}$$

Solution
The coefficients of this Fourier series will be:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}[3\pi - 2\pi] = 1;$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} 3\cos(nt)dt - \int_{0}^{\pi} 2\cos(nt)dt \right]$$

$$= -\frac{3}{n\pi} \sin(nt)\Big|_{-\pi}^{0} - \frac{2}{n\pi} \sin(nt)\Big|_{0}^{\pi} = 0;$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 3\sin(nt)dt - \int_{0}^{\pi} 2\sin(nt)dt\right] \\
&= -\frac{3}{n\pi}\cos(nt)\bigg|_{-\pi}^{0} - \frac{2}{n\pi}\cos(nt)\bigg|_{0}^{\pi} \\
&= -\frac{3}{n\pi} + \frac{3}{n\pi}(-1)^n + \frac{2}{n\pi}((-1)^n - 1)
\end{aligned}
$$

so

$$
b_n = \begin{cases} 0 & even \\ -\frac{10}{n\pi} & odd \end{cases}
$$

Therefore, the Fourier transform of $f(t)$ is:

$$
f(t) \sim \frac{1}{2} - \frac{10}{\pi}\left[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right].
$$