

Math 2280 - Lecture 33

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Today we're going to examine how we find power series solutions to ordinary differential equations of the form

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

Now, as we saw in our last lecture, we're not always able to find a solution to a differential equation of this form. Today we're going to focus on a situation where we know we can find a solution. In particular, around a point a where $P(x) = B(x)/A(x)$, and $Q(x) = C(x)/A(x)$ are analytic. Such a point a is called an *ordinary point*.

Today's lecture corresponds with section 8.2 of the textbook. The assigned problems for this section are

Section 8.2 - 1, 7, 14, 17, 32

Series Solutions Near Ordinary Points

Recall that a function $f(x)$ is analytic at a point $x = a$ if $f(x)$ is equal to a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all values of x around a .

Suppose we have a second-order linear differential equation of the form

$$A(x)y'' + B(x)y' + C(x)y = 0,$$

where $A(x)$, $B(x)$, and $C(x)$ are analytic at $x = a$. If the functions $P(x) = B(x)/A(x)$ and $Q(x) = C(x)/A(x)$ are also analytic at $x = a$, then the differential equation has two linearly independent power series solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n(x - a)^n.$$

The radius of convergence of any such series is at least as large as the distance from a to the nearest (real or complex) singular point of the differential equation.

Note that polynomials are always analytic. If $A(x)$, $B(x)$, and $C(x)$ are polynomials, then $P(x) = B(x)/A(x)$, $Q(x) = C(x)/A(x)$ will be analytic at $x = a$ if $A(a) \neq 0$.

Now, it could be the case that $A(a) = 0$, but $P(x)$ is *still* analytic at $x = a$. For example,

$$P(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

is considered analytic at $x = 0$.¹ But, for most of the problems we'll be dealing with, the functions $A(x)$, $B(x)$, and $C(x)$ will be polynomials, and the only possible problem points are where $A(x) = 0$.

¹The book doesn't do a great job of explaining this. Really, the function $B(x)/A(x)$ would not be defined at $x = a$, and so would *not* be analytic at $x = a$. However, there could be a well-defined value for $P(a)$ that would make $P(x)$ analytic at $x = a$, and so we essentially "fill in" that point. Yeah, it's confusing, and explaining exactly what's going on in detail would take us too far afield, so don't worry too much about it for now. If you take a more advanced class, it will be explained.

Example - Find a general solution in powers of x to the differential equation

$$(x^2 + 2)y'' + 4xy' + 2y = 0.$$

Solution - We assume our solution is analytic of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Differentiating this, and then doing so again, we get:

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}.$$

Plugging these relations into our differential equation gives us:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=2}^{\infty} 2n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0.$$

The x^0 term in the power series on the left will have coefficient $4c_2 + 2c_0$, and so (after we divide by 2) the identity principle gives us

$$2c_2 + c_0 = 0.$$

The x^1 term in the power series will have coefficient $12c_3 + 6c_1$, and so (after we divide by 6) the identity principle gives us

$$2c_3 + c_1 = 0.$$

Now for the x^n terms, where $n \geq 2$, if we shift the second sum above by 2 we get:

$$\sum_{n=2}^{\infty} (n(n-1)c_n + 2(n+2)(n+1)c_{n+2} + 4nc_n + 2c_n)x^n.$$

Again by the identity principle we must have

$$\begin{aligned} n(n-1)c_n + 2(n+2)(n+1)c_{n+2} + 4nc_n + 2c_n = \\ 2(n+2)(n+1)c_{n+2} + (n+2)(n+1)c_n = 0. \end{aligned}$$

From this we get $2c_{n+2} + c_n = 0$ for all $n \geq 2$, which we note also happens to be what we got for $n = 0$ and $n = 1$.

So, for the even coefficients we have

$$\begin{aligned} c_0 &= c_0, \\ c_2 &= -\frac{c_0}{2}, \\ c_4 &= -\frac{c_2}{2} = \frac{c_0}{2^2}, \\ &\vdots \end{aligned}$$

and, in general,²

$$c_{2n} = \frac{(-1)^n}{2^n} c_0.$$

Using identical reasoning we get for the odd terms:

$$c_{2n+1} = \frac{(-1)^n}{2^n} c_1.$$

²You could formally prove this using induction without too much trouble.

From these we get our general solution:

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n} + c_1 \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} x^{2n+1}.$$

Now, at this point we're done, except we should mention if we can what the radius of convergence of this solution is. One way to do this is to note that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n} = \sum_{n=0}^{\infty} \left(\frac{-x^2}{2} \right)^n$$

is a geometric series. It will converge for $\left| \frac{-x^2}{2} \right| < 1$, or in other words for $|x| < \sqrt{2}$. If $|x| < \sqrt{2}$ the above series can be written in closed form as:

$$\sum_{n=0}^{\infty} \left(\frac{-x^2}{2} \right)^n = \frac{1}{1 + \frac{x^2}{2}} = \frac{2}{2 + x^2}.$$

Applying this to both the series solutions above we get:

$$y(x) = \frac{2c_0}{2 + x^2} + \frac{2c_1 x}{2 + x^2},$$

with radius of convergence $|x| < \sqrt{2}$.

We note finally that all our coefficient functions are polynomials, so the only singular points will be where $x^2 + 2 = 0$. This will happen when $x = \pm\sqrt{2}i$. The distance from the origin, $x = 0$, to either of these points is $\sqrt{2}$, so the radius of convergence must be at least $\sqrt{2}$. Turns out it is, in fact, $\sqrt{2}$.

Example - Find a general solution in powers of x to the differential equation

$$y'' + x^2y' + 2xy = 0.$$

Solution - Just as in the previous example we assume our solution is a power series, differentiate it, and plug the series and its derivatives into the differential equation. If we do this we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n+1} + \sum_{n=0}^{\infty} 2c_n x^{n+1} = 0.$$

The x^0 coefficient will be $2c_2$, and by the identity principle we get $2c_2 = 0$.

The x^1 coefficient will be $6c_3 + 2c_0$, and by the identity principle we get $6c_3 + 2c_0 = 0$.

The x^k terms, for $k \geq 2$, will be (shifting the first sum over by 3):

$$\sum_{n=1}^{\infty} ((n+3)(n+2)c_{n+3} + n c_n + 2c_n) x^{n+1}.$$

Again, by the identity principle, we get

$$(n+3)(n+2)c_{n+3} + (n+2)c_n = 0,$$

for $n \geq 1$. The x^1 coefficient tells us this relation is valid even for $n = 0$, and the x^0 coefficient tells us $c_2 = 0$. From these relations we get

$$\begin{aligned} c_0 &= c_0, \\ c_3 &= -\frac{c_0}{3}, \\ c_6 &= -\frac{c_3}{6} = \frac{c_0}{3 \times 6}, \\ &\vdots \end{aligned}$$

and, in general,

$$c_{3n} = \frac{(-1)^n c_0}{3^n n!}.$$

If we apply the same reasoning to the $3n + 1$ class of terms we get

$$c_{3n+1} = \frac{(-1)^n c_1}{1 \times 4 \times 7 \times \cdots \times (3n + 1)}.$$

All the c_{3n+2} terms are 0. So, we get the general solution

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n n!} x^{3n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{1 \times 4 \times \cdots \times (3n + 1)} x^{3n+1}.$$

This solution will be valid (convergent) everywhere. We could prove this without too much bother, or we could just note that all the coefficient functions in our ODE are polynomials, and the leading coefficient function is constant, and so there are no singular points. Therefore, the radius of convergence of the power series solution must be infinite.