

# Math 2280 - Lecture 33

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Today we're going to examine how we find power series solutions to ordinary differential equations of the form

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

Now, as we saw in our last lecture, we're not always able to find a solution to a differential equation of this form. Today we're going to focus on a situation where we know we can find a solution. In particular, around a point  $a$  where  $P(x) = B(x)/A(x)$ , and  $Q(x) = C(x)/A(x)$  are analytic. Such a point  $a$  is called an *ordinary point*.

Today's lecture corresponds with section 8.2 of the textbook. The assigned problems for this section are

Section 8.2 - 1, 7, 14, 17, 32

## Series Solutions Near Ordinary Points

Recall that a function  $f(x)$  is analytic at a point  $x = a$  if  $f(x)$  is equal to a power series  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  for all values of  $x$  around  $a$ .

Suppose we have a second-order linear differential equation of the form

$$A(x)y'' + B(x)y' + C(x)y = 0,$$

where  $A(x)$ ,  $B(x)$ , and  $C(x)$  are analytic at  $x = a$ . If the functions  $P(x) = B(x)/A(x)$  and  $Q(x) = C(x)/A(x)$  are also analytic at  $x = a$ , then the differential equation has two linearly independent power series solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n(x - a)^n.$$

The radius of convergence of any such series is at least as large as the distance from  $a$  to the nearest (real or complex) singular point of the differential equation.

Note that polynomials are always analytic. If  $A(x)$ ,  $B(x)$ , and  $C(x)$  are polynomials, then  $P(x) = B(x)/A(x)$ ,  $Q(x) = C(x)/A(x)$  will be analytic at  $x = a$  if  $A(a) \neq 0$ .

Now, it could be the case that  $A(a) = 0$ , but  $P(x)$  is *still* analytic at  $x = a$ . For example,

$$P(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

is considered analytic at  $x = 0$ .<sup>1</sup> But, for most of the problems we'll be dealing with, the functions  $A(x)$ ,  $B(x)$ , and  $C(x)$  will be polynomials, and the only possible problem points are where  $A(x) = 0$ .

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<sup>1</sup>The book doesn't do a great job of explaining this. Really, the function  $B(x)/A(x)$  would not be defined at  $x = a$ , and so would *not* be analytic at  $x = a$ . However, there could be a well-defined value for  $P(a)$  that would make  $P(x)$  analytic at  $x = a$ , and so we essentially "fill in" that point. Yeah, it's confusing, and explaining exactly what's going on in detail would take us too far afield, so don't worry too much about it for now. If you take a more advanced class, it will be explained.

*Example* - Find a general solution in powers of  $x$  to the differential equation

$$(x^2 + 2)y'' + 4xy' + 2y = 0.$$

More room for the example problem, if you need it.

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$$y'' + x^2 y' + 2xy = 0.$$

More room for the example problem, if you need it.