Math 2280 - Lecture 32

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Up to this point we've focused almost exclusively on solving linear differential equations with constant coefficients. But these are, to say the least, not all the differential equations that are out there. For example, a differential equation that is encountered very frequently is *Bessel's equation* of order n:

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

A very powerful method for solving linear differential equations with variable coefficients is through the use of power series. We'll introduce this method today.

This lecture corresponds with section 8.1 from the textbook. The assigned problems are:

Section 8.1 - 2, 8, 13, 21, 25

Introduction and Review of Power Series

A power series is an infinite series of the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

If $a = 0$ then we call it a power series in x:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 = c_1 x + c_2 x^2 + \cdots$$

We will confine ourselves mainly to power series in x, but every general property of power series in x can be converted to a general property of power series in $(x - a)$.

We say a power series converges on the interval I provided that the limit

$$\sum_{n=0}^{\infty} c_n x^n = \lim_{N \to \infty} \sum_{n=0}^{N} c_n x^n$$

is defined on I. In this case the sum

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

is defined on I, and we call the series a *power series representation* of the function f on I.

Some common power series representations are:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

The first two series converge for all x, while the third, called the geometric series, only converges for $|x| < 1$.

The Power Series Method

The *power series method* for solving a differential equation consists of substituting the power series

$$y = \sum_{n=0}^{\infty} c_n x^n$$

into the differential equation, and then attempting to determine what the coefficients $c_0, c_1, c_2, \ldots$ must be in order for the power series to satisfy the differential equation.

In solving these differential equations, there are two very important theorems:

Theorem - *Termwise Differentiation and Integration of Power Series*

If the power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

of the function f converges on the open interval I, then f is differentiable on I, and

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

at each point of I.

And the other theorem is:

Theorem - *Identity Principle*

If

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for every point x in some open interval I, then $a_n = b_n$ for all $n \geq 0$.

In particular, if $\sum a_n x^n = 0$ for all x in some open interval, it follows from the identity principle that $a_n = 0$ for all $n \geq 0$.

Now, if we have a power series solution to a differential equation, an important question is the interval upon which the series converges. A useful test for determining this interval is the following:

Theorem - *Radius of Convergence*

Given the power series $\sum c_n x^n$, suppose that the limit

$$\rho = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|$$

exists (ρ is finite) or is infinite. Then

(a) If $\rho = 0$, then the series diverges for all $x \neq 0$.

(b) If $0 < \rho < \infty$, then $\sum c_n x^n$ converges if $|x| < \rho$ and diverges if $|x| > \rho$.

(c) If $\rho = \infty$, then the series converges for all x.

The number ρ is called the *radius of convergence* of the power series $\sum c_n x^n$.

Let's see how the power series method works with a few examples.

Example - Solve the differential equation $y' = y$.

Example - Solve the equation $x^2y' = y - x - 1$.

Example - Solve the equation $y'' + y = 0$.

More room for the example.