Math 2280 - Lecture 31

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Today is our final lecture on Laplace transforms, and in a sense, this is really too bad. The reason it's too bad is that we're going to introduce some ideas today that are at the heart of why Laplace transform, and and other transform, methods are so useful. So, in a sense, we're just getting started! If you continue on with differential equations, or even moreso if you take engineering or physics classes that involve solving a lot of differential equations, you'll see the ideas from this lecture again.

This lecture corresponds with section 7.6 from the textbook. The assigned problems are:

Section 7.6 - 1, 6, 11, 14, 15

Impulse and Delta Functions

Consider a force $f(t)$ that acts only during a very short time interval, $a \leq t \leq b$, with $f(t) = 0$ outside this interval. A bat striking a ball or a bolt of lightning striking a tower, for example. Typically, the effect of this force depends only on the integral:

$$p = \int_a^b f(t)dt.$$

This number is called the *impulse* of the force $f(t)$ over the interval $[a, b]$.

An example of this is that the change in momentum of a particle is determined by the impulse of the force acting upon it.

This is nice because frequently we don't know exactly what the force $f(t)$ is, but we can figure out what the integral above, the impulse, is, and it turns out that that's really all we need to know.

Now, if we have a given impulse p, we may as well model it with the simplest function we can, namely, a constant function. So, if we have an impulse $p = 1$, we can get this impulse using the function:

$$d_{a,\epsilon}(t) = \begin{cases} \frac{1}{\epsilon} & a \leq t < a + \epsilon \\ 0 & otherwise \end{cases}$$

where ϵ is the amount of time the impulse acts. We see, if $a > 0$, that

$$\int_0^\infty d_{a,\epsilon}(t)dt = 1.$$

The time interval ϵ over which the impulse acts is frequently *very* small, and it's difficult to get a good measure of what it is. So, we can try to model an *instantaneous impulse* that occurs preciesely at the time $t = a$. We call this instantaneous impulse the *Dirac delta funciton*, and we represent it as:

$$\delta_a(t) = \lim_{\epsilon \to 0} d_{a,\epsilon}(t).$$

Now, this delta function isn't a "function" in the strictest sense. It's 0 everywhere except at the point a, and at a it's infinite. Infinity isn't well defined, and a function that is 0 everywhere except at a point should integrate to 0 over any finite interval. So, what gives? Well, the Dirac delta "function" is actually a generalized function called a distribution, and is only defined in terms of how it operates on integrals.

Delta Functions as Operators

The mean value theorem for integrals states that:

$$\int_a^{a+\epsilon} g(t)dt = \epsilon g(\bar{t})$$

where $\bar{t}$ is a point in $[a, a+\epsilon]$. It follows that:

$$\lim_{\epsilon \to 0} \int_0^{\infty} g(t) d_{a,\epsilon}(t)dt = \lim_{\epsilon \to 0} \int_a^{a+\epsilon} g(t) \cdot \frac{1}{\epsilon} dt = \lim_{\epsilon \to 0} g(\bar{t}) = g(a).$$

We take this as the *definition* of the Delta function. It's an operator such that:

$$\int_0^{\infty} f(t)\delta_a(t)dt = f(a).$$

We note that if $f(t) = e^{-st}$ we get:

$$\int_0^{\infty} e^{-st}\delta_a(t)dt = e^{-as}.$$

We *define* the Laplace transform of the delta function to be:

$$\mathcal{L}(\delta_a(t)) = e^{-as} \ (a \geq 0).$$

If $a = 0$ this gives us:

$$\mathcal{L}(\delta(t)) = 1.$$

Notice as $s \to \infty$ this Laplace transform does *not* go to 0, a further implication that the Delta function is not a standard type of function.

Delta Function Inputs

Suppose we are given a mechanical system whose response $x(t)$ to the external force $f(t)$ is determined by the differential equation:

$$Ax'' + Bx' + Cx = f(t).$$

We want to investigate the response of this system to a unit impulse at the time $t = a$. It seems reasonable to express this response as the solution to the differential equation:

$$Ax'' + Bx' + Cx = \delta_a(t).$$

But, again, $\delta_a(t)$ isn't really a function, and so what would we mean by a solution to the above equation? We call $x(t)$ a solution to the above differential equation provided that:

$$x(t) = \lim_{\epsilon \to 0} x_\epsilon(t),$$

where $x_\epsilon(t)$ is a solution of the differential equation:

$$Ax'' + Bx' + Cx = d_{a,\epsilon}(t).$$

The way to find $x(t)$ is to take the Laplace transform of both sides, figure out $X(s)$, and then figure out its inverse Laplace transform. This is how we solve these types of differential equations, and it's the first major instance we've seen where Laplace transform methods are absolutely necessary.

Example - Solve the initial value problem:

$$x'' + 4x = \delta(t) + \delta(t - \pi);$$
$$x(0) = x'(0) = 0.$$

Solution - If we take the Laplace transform of both sides we get:

$$s^2 X(s) + 4X(s) = 1 + e^{-\pi s},$$

and solving this for $X(s)$ we get:

$$X(s) = \frac{1 + e^{-\pi s}}{s^2 + 4},$$

from which we can calculate the inverse Laplace transform using our table of Laplace transforms:

$$x(t) = \frac{1}{2}\sin(2t) + \frac{1}{2}u(t-\pi)\sin(2(t-\pi)) = \frac{1}{2}\sin(2t)(1 + u(t-\pi)).$$

Example - Solve the initial value problem:

$$x'' + 2x' + x = t + \delta(t);$$

$$x(0) = 0,\ x'(0) = 1.$$

Solution - Taking the Laplace transform of both sides we get:

$$s^2 X(s) - 1 + 2sX(s) + X(s) = \frac{1}{s^2} + 1.$$

Solving for $X(s)$:

$$X(s) = \frac{1}{s^2(s+1)^2} + \frac{2}{(s+1)^2}.$$

Taking a partial fraction decomposition:

$$X(s) = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{3}{(s+1)^2}.$$

This has the inverse Laplace transform:

$$x(t) = -2 + t + 2e^{-t} + 3te^{-t}.$$

Systems Analysis and Duhamel's Principle

Consider a physical system in which the output $x(t)$ to the input function $f(t)$ is described by the differential equation:

$$ax'' + bx' + cx = f(t),$$

where the constant coefficients a, b and c are determined by the physical parameters of the system and are independent of $f(t)$. We assume for simplicity that the system is initially passive, and so $x(0) = x'(0) = 0$. The Laplace transform of the differential equation is:

$$as^2X(s) + bsX(s) + cX(s) = F(s),$$

and so $X(s)$ is:

$$X(s) = \frac{F(s)}{as^2 + bs + c} = W(s)F(s).$$

Here the function

$$W(s) = \frac{1}{as^2 + bs + c}$$

is called the *transfer function* of the system. The inverse Laplace transform of the transform function, $w(t)$, is called the *weight function* of the system. Using our earlier results about convolutions and the above formula for $X(s)$, we get that the solution to our system is:

$$x(t) = \int_0^t w(\tau)f(t-\tau)dt.$$

This is called *Duhamel's principle* for the system, and the important thing about it is that the weight function $w(t)$ is determined completely by the parameters of the system, and has nothing to do with the imput function $f(t)$. So, if we know the weight function, we can calculate the solution for *any* input by "just" calculting an integral. Now, integrals aren't easy, but they're easier than solving differential equations. It's interesting (actually, it's *very* interesting, for reasons we won't explore in this class) that our weight function is actually the response of our system to a delta function input.

Example - Apply Duhamel's principle to write an integral formula for the solution of the initial value problem:

$$x'' + 6x' + 9x = f(t);$$
$$x(0) = x'(0) = 0.$$

Solution - The transfer function of this system is:

$$X(s) = \frac{1}{s^2 + 6s + 9} = \frac{1}{(s+3)^2}.$$

The inverse Laplace transform of this transfer function, the weight function, will be:

$$w(t) = te^{-3t}.$$

So, the response $x(t)$ will be given by the integral equation:

$$x(t) = \int_0^t \tau e^{-3\tau} f(t-\tau)d\tau.$$