

Math 2280 - Lecture 2: How To Use Integration To Solve Differential Equations

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The major topic of calculus II was the study of integrals and how we integrate functions. Well, it turns out that finding an integral is actually solving a special type of differential equation.

We're going to talk today about the form of these differential equations, how they come up, and how we can extract some more general ideas about differential equations based upon these (relatively) simple ones.

Integrate Your Way to a Solution

If we have a first-order differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

then it takes a particularly simple form if $f(x, y)$ can be written as just a function of x , so $f(x)$. Then we have an equation of the form

$$\frac{dy}{dx} = f(x).$$

To solve this for $y(x)$ we just need to integrate both sides of the equation

$$y(x) = F(x) + C$$

where C is an arbitrary constant and $F(x)$ is a function such that $F'(x) = f(x)$. We usually choose $F(x)$ so that it has no constant term.

If all we're given is a differential equation, then this is as far as we can go. However, if we're given the value of $y(x)$ as a point, say $y(x_0) = y_0$, then we can solve for C :

$$C = y_0 - F(x_0)$$

to get a *particular solution* to our differential equation. It's the one and only solution satisfying our differential equation and $y(x_0) = y_0$.

Example - Find a function satisfying the differential equation

$$\frac{dy}{dx} = (x - 2)^2$$

with the initial condition $y(2) = 1$.

$$dy = (x - 2)^2 dx$$

$$y = \int (x - 2)^2 dx = \frac{(x - 2)^3}{3} + C$$

$$y(2) = \frac{(2 - 2)^2}{3} + C = 0 + C = C = 1$$

So,

$$y(x) = \frac{(x - 2)^3}{3} + 1$$

If we have a second-order differential equation of the form

$$\frac{d^2y}{dx^2} = f(x)$$

then if we integrate both sides of the equation once we get

$$\frac{dy}{dx} = F(x) + C_1,$$

and if we integrate again we get

$$y(x) = \int F(x)dx + C_1x + C_2.$$

This solution has *two* unknown constants. If we know $y(x_0)$ and $y'(x_0)$ then we can solve for C_1 and C_2 . So, for a second-order differential equation there are two unknown constants, and we need two initial conditions.

This is a general phenomenon. We'll find that, generally speaking, the solution to an n -th order differential equation will require n initial conditions to specify a particular solution.

Example - What is the solution to the differential equation

$$\frac{d^2y}{dx^2} = 3$$

if $y'(2) = 4$ and $y(2) = 7$?

$$\frac{dy}{dx} = 3x + C_1 \quad y'(2) = 3(2) + C_1 = 4 \Rightarrow C_1 = -2$$

$$\frac{dy}{dx} = 3x - 2 \quad y(x) = \frac{3}{2}x^2 - 2x + C_2$$

$$y(2) = \frac{3}{2}(2^2) - 2(2) + C_2 = 6 - 4 + C_2 = 7 \Rightarrow C_2 = 5$$

$$\boxed{y(x) = \frac{3}{2}x^2 - 2x + 5}$$

Velocity and Acceleration

Newton's second law of motion states that the net force upon an object is equal to the object's mass multiplied by its acceleration. Acceleration is the rate of change of velocity, which is itself the rate of change of position.

So, if we represent the acceleration of an object at time t as $a(t)$, the velocity at time t as $v(t)$, and the position at time t as $x(t)$, then we have the relations;

$$\frac{dx}{dt} = v(t),$$

$$\frac{dv}{dt} = a(t),$$

and therefore

$$\frac{d^2x}{dt^2} = a(t).$$

So, acceleration is the second derivative of position with respect to time.

If an object is experiencing constant acceleration a then

$$a(t) = a$$

and therefore its velocity will be

$$v(t) = \int a = at + C_1.$$

If the velocity at time $t = 0$ is v_0 , then we have

$$v(t) = at + v_0.$$

If we then integrate this again to get position we get

$$x(t) = \frac{1}{2}at^2 + v_0t + C_2.$$

If the position at time $t = 0$ is x_0 , then our position function is

$$x(t) = \frac{1}{2}at^2 + v_0t + x_0.$$

Example - At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point C , 35 miles away. If the constantly accelerated car arrives at C with a velocity of 60 mi/h, at what time does it arrive at C ?

$$v(t) = at + v_0$$

$$v_0 = 0$$

$$\Rightarrow v(t) = at$$

t_f = total time

$$v(t_f) = at_f = 60$$

$$\Rightarrow a = \frac{60}{t_f}$$

$$x(t) = \frac{1}{2}at^2 + v_0t + x_0$$

$$x_0 = 0, v_0 = 0$$

$$x(t) = \frac{1}{2}at^2$$

$$x(t_f) = \frac{1}{2} \left(\frac{60}{t_f} \right) (t_f^2) = 35$$

$$30t_f = 35$$

$$t_f = \frac{35}{30} = \frac{7}{6} \text{ hours}$$

$$\text{Noon} + \frac{7}{6} \text{ hours}$$

$$= \boxed{1:10 \text{ PM}}$$