Math 2280 - Lecture 24

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If we think back to calculus II we'll remember that one of the most important things we learned about in the second half of the course were Taylor series. A Taylor series is a way of expressing a function as an "infinite polynomial". The Taylor series (or in the cases below the Maclaurin series, which is just the Taylor series expanded around the point $x = 0$) for some common functions are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

We're going to use this conception of an exponential function to define what it means to take a "matrix exponential", which is a matrix-valued function of the form:

$$e^{\mathbf{A}}$$

where $\mathbf{A}$ is a constant square matrix. Now, it may be that you've never seen this before, and it's not immediately clear what this means. How do you take something to a matrix power? Well, the way that we define this exponential is in terms of an infinite series:

$$e^{\mathbf{A}} = I + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots$$

Now, we'll say that the rigor police are off the beat here and won't go through a proof that this series always converges, but it's true that this series always converges to some constant matrix, where we view convergence of a series of matrices in terms of convergence of their individual entries.

We're going to see that matrix exponentials have quite a bit to do with finding solutions to first-order linear homogeneous systems with constant coefficients.

This lecture corresponds with section 5.5 of the textbook. The assigned problems for this section are:

Section 5.5 - 1, 7, 9, 18, 24

Fundamental Matrices for Systems of ODEs

To review, a solution to a system of ODEs satisfies the relation:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

If we have n *linearly independent* solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ then *any* solution to our system can be written in the form:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

From these linearly independent solutions we can construct a matrix $\Phi(t)$:

$$\Phi(t) = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$$

which is called a *fundamental matrix* for the system defined by the constant matrix $\mathbf{A}$. Now, our statement that we can write any solution $\mathbf{x}$ as a linear combination of our linearly independent solutions can be written in matrix form as:

$$\mathbf{x} = \Phi(t)\mathbf{c}$$

where **c** is a constant vector given by:

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

This fundamental matrix $\Phi(t)$ is not unique, and if $\Psi(t)$ is another fundamental matrix then we have:

$$\Psi(t) = \begin{pmatrix} \tilde{\mathbf{x}}_1 & \tilde{\mathbf{x}}_2 & \cdots & \tilde{\mathbf{x}}_n \end{pmatrix},$$

where the $\tilde{\mathbf{x}}_i$ are linearly independent solutions to our system of ODEs. Now, each of these solutions can in turn be represented as a linear combination of our original $\mathbf{x}_i$:

$$\begin{aligned} \tilde{\mathbf{x}}_1 &= c_{11}\mathbf{x}_1 + c_{12}\mathbf{x}_2 + \cdots + c_{1n}\mathbf{x}_n \\ \tilde{\mathbf{x}}_2 &= c_{21}\mathbf{x}_1 + c_{22}\mathbf{x}_2 + \cdots + c_{2n}\mathbf{x}_n \\ &\vdots \\ \tilde{\mathbf{x}}_n &= c_{n1}\mathbf{x}_1 + c_{n2}\mathbf{x}_2 + \cdots + c_{nn}\mathbf{x}_n \end{aligned}$$

or, in matrix terminology:

$$\Psi(t) = \Phi(t)\mathbf{C}$$

where

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{pmatrix}.$$

If we have an initial condition:

$$\mathbf{x}(0) = \mathbf{x}_0$$

then we get for our solution:

$$\mathbf{x}_0 = \Psi(0)\mathbf{c}$$

where **c** is a constant vector given by:

$$\mathbf{c} = \Psi(0)^{-1}\mathbf{x}_0$$

and we note that $\Psi(0)^{-1}$ makes sense as $\Psi(t)$ is nonsingular for all t by definition.

Combining these results we get:

$$\mathbf{x}(t) = \Psi(t)\Psi(0)^{-1}\mathbf{x}_0.$$

Example - Find a fundamental matrix for the system below, and then find a solution satisfying the given initial conditions.

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \mathbf{x}$$

with initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

More room for the example...

Now, this may seem like a more difficult way of figuring out the same solutions we'd figured out using other methods and, well, that's because it is. However, the major advantage to this method is that it makes it very easy to switch around our initial conditions. Once we've found $\Phi(t)$ and $\Phi(0)^{-1}$, it becomes very easy to find the solution for any given initial conditions $\mathbf{x}_0$ just using the relation $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$.

Matrix Exponentials and ODEs

Right about now you might be asking why we discussed matrix exponentials at the beginning of this lecture. We haven't used them yet. Well, now is when we're going to talk about them, and we'll see they are intimately connected to fundamental matrices.

First, let's go over some important properties of matrix exponentials.

If $\mathbf{A}$ is a diagonal matrix:

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

then if we exponentiate this matrix we get:

$$e^{\mathbf{A}} = \begin{pmatrix} e^a & 0 & 0 \\ 0 & e^b & 0 \\ 0 & 0 & e^c \end{pmatrix}.$$

Of course the same idea works, mutatis mutandis, for a diagonal matrix of any size.

Also, if two matrices commute, so $\mathbf{AB} = \mathbf{BA}$, then:

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}.$$

Note this this is *not* necessarily true if $\mathbf{A}$ and $\mathbf{B}$ do not commute.

Finally, if we have the matrix exponential:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots$$

then we can differentiate these term by term[1] and we get:

$$\frac{de^{\mathbf{A}t}}{dt} = \mathbf{A} + \mathbf{A}^2 t + \frac{\mathbf{A}^3}{2!}t^2 + \cdots = \mathbf{A}e^{\mathbf{A}t}.$$

Well, this is *very* interesting. What this is saying is that the matrix $e^{\mathbf{A}t}$ satisfies the matrix differential equation:

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

where $\mathbf{X}$ is a square matrix the same size as $\mathbf{A}$, and each column of $\mathbf{x}$ satisfies $\mathbf{x}_i' = \mathbf{A}\mathbf{x}_i$. Now, as $e^{\mathbf{A}t}$ is nonsingular, each of its columns must be linearly independent, and so $e^{\mathbf{A}t}$ is a fundamental matrix for $\mathbf{A}$!

If we note finally that $e^{\mathbf{A}0} = \mathbf{I} = \mathbf{I}^{-1}$ then we get the relation:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0.$$

where $\mathbf{x}(t)$ is a solution to the system of ODEs:

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

with initial condition:

$$\mathbf{x}(0) = \mathbf{x}_0.$$

Note this is *very* similar to the solution of the differential equation:

$$x' = x.$$

[1] Again, just trust me, the rigor politce are off duty here...

Example - Solve the system of differential equations:

$$\mathbf{x}' = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$$

with initial conditions:

$$\mathbf{x}(0) = \begin{pmatrix} 19 \\ 29 \\ 39 \end{pmatrix}.$$

More room for the example...