Math 2280 - Lecture 21

Dylan Zwick

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First note that we're a bit behind schedule, and as I've chosen not to focus on numerical methods in this class we'll be skipping section 4.3 from the textbook. So, today we'll begin chapter 5 on linear systems of differential equations.

Today's lecture will be mostly a review of linear algebra. Please note that this is intended to be a *review*, and so the first 80% of the lecture should be familiar to you. If it's not, please try to review the material ASAP.

The assigned problems for the corresponding section from the textbook are:

Section 5.1 - 1, 7, 15, 21, 27

A Review of Linear Algebra

The fundamental object we study in linear algebra is the matrix, which is a rectangular array of numbers:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

This is an $m \times n$ matrix, which means that it has m rows and n columns. We note that the number of rows and number of columns do not have to

be the same, although frequently for the matrices we deal with they will be.

Basic Operations

Addition of matrices is termwise. We can only add two matrices if they are the same size. Multiplying a matrix by a constant multiplies every element in the matrix by that constant.

Example - Calculate the following sum and product:

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 9 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 8 & 13 \end{pmatrix}$$

$$4 \begin{pmatrix} 2 & 9 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 8 & 36 \\ 20 & 24 \end{pmatrix}$$

Matrix addition and scalar multiplication of matrices satisfy the following basic properties:

1. $\mathbf{A} + \mathbf{0} = \mathbf{A}$ (additive identity)

2. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (commutativity of addition)

3. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ (associativity of addition)

4. $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$ (distributivity over matrix addition)

5. $(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$ (distributivity over scalar addition)

where here bold faced capital[1] letters represent matrices, lower case standard letters represent scalars, and $\mathbf{0}$ represents the zero matrix of the appropriate size[2].

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[1]Did you know that a word for a capital letter is a "majascule", and a word for a lower-case letter is a "miniscule"?

[2]The zero matrix is a matrix with all zero entries.

Dot Products and Matrix Multiplication

The $1 \times n$ matrices are frequently called vectors or column vectors, while the $m \times 1$ matrices are frequently called row vectors. We can take the dot product of two vectors (which can be formulated as the product of a row vector with a column vector) by summing the products of the entries on the two vectors

Example - Suppose $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix}$, calculate $\mathbf{a} \cdot \mathbf{b}$.

$$\mathbf{a} \cdot \mathbf{b} = 3 \times 2 + 4 \times 9 + 7 \times 6 = 84.$$

Matrices can be multiplied as well as summed. The basic formula for matrix multiplication is:

$$\mathbf{AB}_{ij} = \sum_{r=1}^{k} a_{ik} b_{kj}$$

where the matrix $\mathbf{AB}$ is the matrix $\mathbf{A}$ right multiplied by the matrix $\mathbf{B}$. We note that for this to work out the number of columns in $\mathbf{A}$ must equal the number of rows in $\mathbf{B}$. We can view the entry $\mathbf{AB}_{ij}$ as being the dot product of the ith row of matrix $\mathbf{A}$ with the jth column of matrix $\mathbf{B}$.

Example - For the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 & 4 \\ 1 & 5 \end{pmatrix},$$

calculate $\mathbf{AB}$ and $\mathbf{BA}$.

$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 23 \\ 13 & 21 \end{pmatrix},$$

$$\mathbf{BA} = \begin{pmatrix} 3 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 22 & 13 \\ 22 & 8 \end{pmatrix}.$$

Note that, as the examples above illustrate, matrix multiplication is in general *not* commutative. In fact, sometimes when **AB** makes sense **BA** does not! And it's not always the case that **AB** = **BA** even when this makes sense (i.e. they're both square).

Inverses

For square matrices we can talk about inverses. For a square matrix **A**, its inverse (if it exists) is the unique matrix $\mathbf{A}^{-1}$ such that:

$$\mathbf{AA}^{-1} = \mathbf{I}$$

where the matrix **I** is the *identity matrix* that is a square matrix with the entry 1 down the diagonal and the entry 0 everywhere else. The identity matrix is called this because if you multiply it with any other appropriate (same size) matrix you just get back that matrix. We again stress that these concepts only make sense for square matrices.

Determinants

For a 2×2 matrix the *determinant* is defined as:

$$det(\mathbf{A}) = det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Higher-order determinants can be calculated by row or column expansion.

Example - Calculate

$$\begin{vmatrix} 3 & 2 & 1 \\ 5 & 6 & 2 \\ 0 & 1 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 2 & 1 \\ 5 & 6 & 2 \\ 0 & 1 & 4 \end{vmatrix} = 3 \begin{vmatrix} 6 & 2 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 5 & 2 \\ 0 & 4 \end{vmatrix} + 1 \begin{vmatrix} 5 & 6 \\ 0 & 1 \end{vmatrix} = 31$$

We note that we'd get the same value if we chose to expand along any row or column.

Now, a big (in some sense *the* big) theorem from linear algebra states that a square matrix is invertible if and only if it has a non-zero determinant. Note that the concept of determinant as such doesn't make sense for non-square matrices.

Matrix-Valued Functions

The entries of a vector, or matrix for that matter, don't have to be constants, and they can even be functions. A vector-valued function is a vector all of whose components are functions:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

where the $x_i(t)$ are standard (scalar-valued) functions.

In a similar way we can define matrix-valued functions:

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

We can make sense of the derivative of vector or matrix valued functions just by defining it as the derivative of each entry termwise. If we define differentiation of matrix-valued functions this way we recover a form of the product rule:

$$\frac{d(\mathbf{AB})}{dt} = \frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt}.$$

We note that these terms are all matrices, and matrix multiplication is in general *not* commutative, so don't switch these terms around even if they're square and you can!

Example - Calculate $\mathbf{A}'(t)$ for the matrix:

$$\mathbf{A}(t) = \begin{pmatrix} e^t & t^2 - 2t \\ \cos{(4t)} & 3^t + 1 \end{pmatrix}$$

$$\mathbf{A}'(t) = \begin{pmatrix} e^t & 2t - 2 \\ -4\sin{(4t)} & 3^t \ln 3 \end{pmatrix}$$

First-Order Linear Systems

As mentioned in section 4.2, if we have a first-order linear system of differential equations:

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t)$$
$$x_2' = p_{21}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t)$$
$$\cdots$$
$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t)$$

we can express this in a much simpler form as a matrix equation:

$$\frac{d\mathbf{x}}{dt} = \mathbf{Px} + \mathbf{f},$$

where $\mathbf{x}_{i1} = x_i(t)$, $\mathbf{P}_{ij} = p_{ij}(t)$, and $\mathbf{f}_{i1} = f_i(t)$.

A solution to this system of differential equations is a vector valued function $\mathbf{x}$ that satisfies the above differential equation. Now, we call such an equation *homogeneous* if, you guessed it, $\mathbf{f} = \mathbf{0}$.

Superposition

Suppose we have a homogeneous differential equation in the form above and we have solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then any other vector-valued function of the form:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

is also a solution. This follows immediately from the linearity properties of, well, *linear* algebra.

Now, it turns out (not too surprisingly) that *any* solution of a homogeneous differential equation in the form mentioned above can be written as a linear combination of n *linearly independent* solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

We can determine if n solution vectors are linearly independent by checking their (surprise!) Wronskian:

$$W(\mathbf{x}_1, \ldots, \mathbf{x}_n) = det([\mathbf{x}_1 \cdots \mathbf{x}_n]).$$

Example - First verify that the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system.

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}_1 = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$$

Checking the first solutions we have

$$\mathbf{x}_1' = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix},$$

while

$$\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}.$$

So, $\mathbf{x}_1$ is a solution. Checking the other solution we get

$$\mathbf{x}_2' = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix},$$

while

$$\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix}.$$

So, $\mathbf{x}_2$ is also a solution.

The Wronskian of $\mathbf{x}_1$ and $\mathbf{x}_2$ is:

$$W(\mathbf{x}_1, \mathbf{x}_2) = \begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = -2e^{3t} + 3e^{3t} = e^{3t} \neq 0.$$

So, the solutions are linearly independent.

The general solution of the system will be:

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix},$$

where c_1 and c_2 are unknown constants that would be determined by initial conditions.