Math 2280 - Lecture 20

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Today we'll learn about a method for solving systems of differential equations, the *method of elimination*, that is very similar to the elimination methods we learned about in linear algebra. We'll extend this analogy further by learning about polynomial differential operators, and how we can apply analogues of Cramer's rule using these differential operators to solving systems of differential equations.

This lectures corresponds with section 4.2 of the textbook. The assigned problems for this section are:

Section 4.2 - 1, 10, 19, 28, 30

The Method of Elimination

For systems of differential equations, particularly linear systems, we can combine equations like we do in linear algebra to eliminate dependent variables. This simplifies the equation, and eventually can be used to reduce the equation to a single linear differential equations (usually not of first-order) that we can then solve using the methods from chapter 3.

Example - Use the method of elimination to find the solution to the initial value problem:

$$x' = -3x + 2y \quad y' = -3x + 4y;$$
$$x(0) = 1, y(0) = -1$$

More room for the example problem.

$x' = -3x + 2y$
$y' = -3x + 4y$ $\Rightarrow y' - x' = 2y \Rightarrow + x' = y' - 2y$ Yikes!

$\Rightarrow y' - 2y = -3x + 2y \xrightarrow{\text{Differentiate Both Sides}} y'' - 2y' = -3x' + 2y'$

$\Rightarrow y'' - 4y' = -3(y' - 2y) \Rightarrow y'' - y' - 6y = 0$

$r^2 - r - 6 = (r-3)(r+2) \Rightarrow y(t) = c_1 e^{3t} + c_2 e^{-2t}$

$y(t) = c_1 e^{3t} + c_2 e^{-2t}$

$y'(t) = 3c_1 e^{3t} - 2c_2 e^{-2t} = -3x + 4c_1 e^{3t} + 4c_2 e^{-2t}$

$\Rightarrow -3x(t) = -c_1 e^{3t} - 6c_2 e^{-2t} \Rightarrow x(t) = \frac{c_1}{3} e^{3t} + 2c_2 e^{-2t}$

$y(0) = c_1 + c_2 = -1 \Rightarrow c_1 + c_2 = -1 \qquad 5c_2 = 4 \Rightarrow c_2 = \frac{4}{5}$
$x(0) = \frac{c_1}{3} + 2c_2 = 1 \qquad c_1 + 6c_2 = 3 \Rightarrow c_1 = -\frac{9}{5}$ So,

$$\boxed{\begin{aligned} x(t) &= -\frac{3}{5} e^{3t} + \frac{8}{5} e^{-2t} \\ y(t) &= -\frac{9}{5} e^{3t} + \frac{4}{5} e^{-2t} \end{aligned}}$$

Polynomial Differential Operators

A polynomial differential operator is a map from functions to functions of the form:

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0,$$

where D represents the derivative operator. So, for example, the differential operator

$$L = D^2 + 2D - 3$$

when applied to the function

$$f(x) = x^2 + 4x - 5$$

yields

$$L(f) = D^2(x^2 + 4x - 5) + 2D(x^2 + 4x - 5) - 3(x^2 + 4x - 5)$$
$$= (2) + (4x + 8) - (3x^2 + 12x - 15) = -3x^2 - 8x + 25.$$

Polynomial differential operators commute. So, if we have two differential operators, L_1 and L_2, then $L_2 L_1(f) = L_1 L_2(f)$. Note that this is not generally the case for all differential operators. It's not even necessarily the case for polynomial operators with variable coefficients.

Any system of two linear differential equations with constant coefficients can be written in the form

$$\begin{array}{ccccc} L_1 x & + & L_2 y & = & f_1(t) \\ L_3 x & + & L_4 y & = & f_2(t) \end{array}$$

If we act on the top row with the operator L_3, and on the bottom row with the operator L_1 we get

$$\begin{array}{ccccc} L_3 L_1 x & + & L_3 L_2 y & = & L_3 f_1(t) \\ L_1 L_3 x & + & L_1 L_4 y & = & L_1 f_2(t) \end{array}.$$

If we then subtract the first equation from the second, using the fact that the operators commute, we get

$$(L_1 L_4 - L_2 L_3) y = L_1 f_2 - L_3 f_1,$$

in the single variable y. Alternatively, we could have eliminated y in a like manner from the original system and obtained the equation

$$(L_1 L_4 - L_2 L_3) x = L_4 f_1 - L_2 f_2.$$

Note that the same operator, $(L_1L_4 - L_2L_3)$, appears on the left hand side of both equations. This operator is called the operational determinant:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} = L_1L_4 - L_2L_3,$$

and the above two equalities are the operational versions of Cramer's rule:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} f_1 & L_2 \\ f_2 & L_4 \end{vmatrix},$$

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & f_1 \\ L_3 & f_2 \end{vmatrix}.$$

Note that if the operational determinant is identically zero there can be either no solutions, or infinitely many solutions, to the ODE. Also, this approach that we've applied here for systems with two variables generalizes to systems with an arbitrary numbers of variables, although the computations involved for Cramer's rule become more and more narsty.

Example - Show that the following system is degenerate, and then determine whether it has infinitely many or no solutions.

$$\begin{aligned} (D+2)x + (D+2)y &= e^{-3t} \\ (D+3)x + (D+3)y &= e^{-2t} \end{aligned}$$

$(D+2)(D+3) - (D+2)(D+3) = 0$ So, degenerate.

Subtract equation 1 from equation 2 to get:

$x + y = e^{-2t} - e^{-3t}$

$\Rightarrow \boxed{y = e^{-2t} - e^{-3t} - x}$

So, for <u>any</u> function $x(t)$ we can find a $y(t)$ that satisfies this system. So, infinitely many solutions

Example - Find the general solution to the system of differential equations:

$$\begin{array}{rcrcl} (D^2+D)x & + & D^2y & = & 2e^{-t} \\ (D^2-1)x & + & (D^2-D)y & = & 0 \end{array}$$

$$(D^2+D)(D^2-D) - D^2(D^2-1) = D^4 - D^3 + D^3 - D^2 - D^4 + D^2 = 0.$$

So, again, the system is degenerate.

If we subtract the second equation from the first we get:

$$(D+1)x + Dy = 2e^{-t}$$

Differentiating both sides we get:

$$(D^2+D)x + D^2y = -2e^{-t}$$

But, the first equation is

$$(D^2+D)x + D^2y = 2e^{-t}$$

As $2e^{-t} \neq -2e^{-t}$

there are <u>no solutions</u>.