

Math 2280 - Lecture 19

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Up to now all the differential equations with which we've dealt have had one dependent variable and one independent variable. So, a differential equation like:

$$y'' + 2xy' + 3e^x y = \sin x,$$

has independent variable x and dependent variable y . Today, we're going to move on to talking about *systems* of differential equations, in which there are more than one differential equation that must be satisfied, and more than one dependent variable. We will restrict our attention to systems in which the number of equations is the same as the number of dependent variables. So, for example, the system:

$$x' = y$$

$$y' = -x$$

where both x and y are functions of the independent variable t .

Today's lecture corresponds with section 4.1 of the textbook, and the assigned homework problems are:

Section 4.1 - 1, 3, 13, 15, 22

Systems of Equations

If we think back to linear algebra one of the major aims was solving systems of linear equations like this one:

$$\begin{aligned}2x + 3y &= 7 \\3x - 2y &= 4\end{aligned}$$

and a solution to this system were values for x and y that satisfied *both* equations. In this case the unique solution would be $x = 2$ and $y = 1$.

For a system of differential equations we again have multiple equations that must be satisfied. For example:

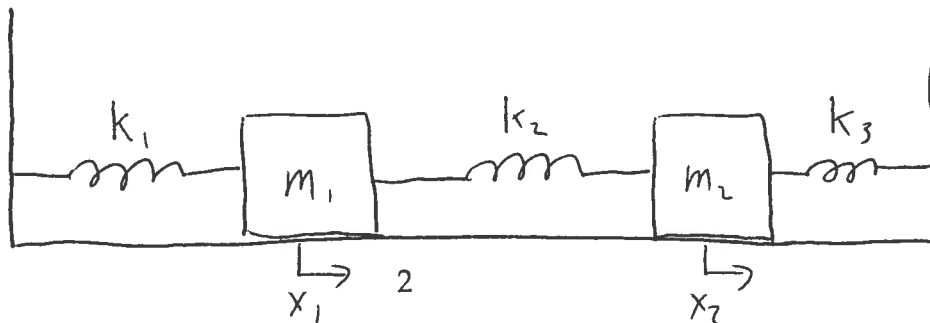
$$\begin{aligned}x' - y &= 0 \\y' + x &= 0\end{aligned}$$

and a solution to this system are two *functions* of the independent variable, $x(t)$ and $y(t)$, that satisfy both equations. In this case $x(t) = A \sin t + B \cos t$ and $y(t) = A \cos t - B \sin t$. Note that there are two unknown constants in this solution, A and B , and they are undetermined unless we're given values of x and y at a point a . So, we need initial conditions to determine a unique solution, but all solutions will be sums of sines and cosines.

Example - Derive the equations

$$\begin{aligned}m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 \\m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2\end{aligned}$$

for the displacements (from equilibrium) of the two masses shown below.



More room for example.

The total force on m_1 will be, from

$$\begin{aligned}\text{Hooke's law: } & -k_1 x_1 - k_2 x_1 + k_2 x_2 \\ & = -k_1 x_1 - k_2 (x_1 - x_2) \\ & = -(k_1 + k_2) x_1 + k_2 x_2\end{aligned}$$

And similarly the force on m_2 will be

$$\begin{aligned}& k_2 x_1 - (k_2 + k_3) x_2 \\ \text{Newton's second law says we have the relation} \\ m_1 x_1'' & = -(k_1 + k_2) x_1 + k_2 x_2 \quad m_2 x_2'' = k_2 x_1 - (k_2 + k_3) x_2\end{aligned}$$

First-Order Systems

A first-order system of differential equations is one where there are n differential equations and n dependent variables, and each equation expresses the derivative of one of the dependent variables in terms of the dependent variables and the independent variable. So, a system with independent variable t and dependent variables x_1, \dots, x_n of the form:

$$\begin{aligned}x_1' &= f_1(t, x_1, x_2, \dots, x_{n-1}, x_n) \\ x_2' &= f_2(t, x_1, x_2, \dots, x_{n-1}, x_n) \\ &\vdots \\ x_{n-1}' &= f_{n-1}(t, x_1, x_2, \dots, x_{n-1}, x_n) \\ x_n' &= f_n(t, x_1, x_2, \dots, x_{n-1}, x_n)\end{aligned}$$

Note that every equation has the derivative of one of the dependent variables on the left, and that's the only derivative in the equation.

First, note that we can frequently transform a higher-order differential equation into a system of first-order differential equations. For example, the differential equation:

$$x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$$

we can rewrite as a system of three first-order equations by defining the variables $x_1 = x, x_2 = x_1', x_3 = x_2' = x_1''$. Then we get the system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 5x_1 - 2x_2 - 3x_3 + \sin 2t \end{aligned}$$

Almost seems like cheating, doesn't it? But, this idea is of real theoretical and practical importance. For example, remember Euler's method? Well, it only applied to first-order equations. Using this idea we can convert a higher order equation into a series of first-order equations, and apply variations of Euler's method!

Example - Transform the differential equation

$$x^{(4)} + 6x'' - 3x' + x = \cos 3t$$

into an equivalent system of first-order differential equations.

$$\begin{aligned} x_1 &= x \\ x_2 &= x_1' \\ x_3 &= x_2' = x_1'' \\ x_4 &= x_3' = x_2'' = x_1''' \\ x_4' &= x_3'' = x_2''' = x_1^{(4)} = -6x_1'' + 3x_1' - x_1 + \cos(3t) \\ &= -6x_3 + 3x_2 - x_1 + \cos(3t) \end{aligned}$$

So,

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= -6x_3 + 3x_2 - x_1 + \cos(3t) \end{aligned}}$$

Linear Systems

In this class so far we've focused a lot of attention on linear differential equations. The main reason for this is that they're much more simple, and easy to solve, than non-linear differential equations, and so we learn them first. Also, many real world applications have linear approximations that can be useful. Systems of differential equations are the same. A *linear* first-order system of differential equations is a system of equations of the form:

$$\begin{aligned}x'_1 &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t), \\x'_2 &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t), \\&\vdots \\x'_n &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t).\end{aligned}$$

All the dependent variables on the right side appear linearly. We can rewrite this suggestively in a matrix format:

$$\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Or, even more compactly using vectors:

$$\mathbf{x}' = \mathbf{P}\mathbf{x} + \mathbf{f}.$$

Might linear algebra techniques be useful for solving systems of differential equations? We'll have to wait and see.¹

Finally, the practical and theoretical foundations of our study of linear differential equations are the various existence and uniqueness theorems. Well, we're in luck, as there is a similar theorem for systems of linear differential equations.

¹OK, fine, the answer is yes.

Theorem - Suppose that the functions $p_{11}, p_{12}, \dots, p_{nn}$ and the functions f_1, f_2, \dots, f_n are continuous on the open interval I containing the point a . Then given the n numbers b_1, b_2, \dots, b_n , the system of differential equations above has a unique solution on the entire interval I that satisfies the n initial conditions

$$x_1(a) = b_1, x_2(a) = b_2, \dots, x_n(a) = b_n.$$

Example - Calculate the unique solution to the given initial value problem:

$$\begin{aligned} x' &= -y \\ y' &= 13x + 4y \\ x(0) &= 0, y(0) = 3. \end{aligned}$$

$$\begin{aligned} y &= -x' \Rightarrow y' = -x'' = 13x - 4x' \\ \Rightarrow x'' - 4x' + 13x &= 0 \end{aligned}$$

$$\frac{4 \pm \sqrt{(4)^2 - 4(1)(13)}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= 2 \pm 3i$$

$s_0,$

$$x(t) = A e^{2t} \cos(3t) + B e^{2t} \sin(3t)$$

$$\begin{aligned} y(t) &= -x'(t) = 3A e^{2t} \sin(3t) - 2A e^{2t} \cos(3t) \\ &\quad - 3B e^{2t} \cos(3t) - 2B e^{2t} \sin(3t) \\ &= (3A - 2B) e^{2t} \sin(3t) - (2A + 3B) e^{2t} \cos(3t) \end{aligned}$$

$$x(0) = A = 0$$

$$y(0) = -3B = 3 \Rightarrow B = -1$$

$s_0,$

$$\boxed{\begin{aligned} x(t) &= -e^{2t} \sin(3t) \\ y(t) &= 3e^{2t} \cos(3t) + 2e^{2t} \sin(3t) \end{aligned}}$$