Math 2280 - Lecture 17

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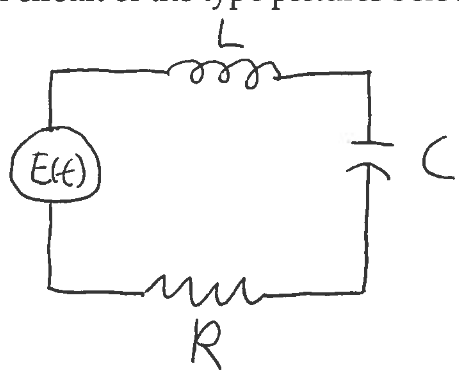
In today's lecture we'll talk about another *very* common physical system that comes up all the time in engineering - a closed circuit with a resistor, a capacitor, and an inductor. We'll learn that, even though physically this system is very different than a mass on a spring, the differential equation that describes them is, essentially, the same. Well, the same equations have *the same* solutions, and we'll see that the solutions we determined for the mass-spring system have their exact analogs for circuits.

Today's lecture corresponds with section 3.7 of the textbook. The assigned problems are:

Section 3.7 - 1, 5, 10, 17, 19

Electrical Circuits

For an electrical circuit of the type pictures below:



Kirchoff's second law tells us that the sum of the voltage drops across each component must equal 0:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t).$$

This is a second order linear ODE with constant coefficients! So everybody chill out, we've got this. For example, if

$$E(t) = E_0 \sin{(\omega t)},$$

if we differentiate both sides of the equation we get:

$$LI'' + RI' + \frac{1}{C}I = \omega E_0 \cos \omega t.$$

The homogeneous solution to this will be:

$$y_h = e^{-\frac{Rt}{2L}} \left(c_1 e^{\frac{\sqrt{R^2-4L/C}}{2L}t} + c_2 e^{-\frac{\sqrt{R^2-4L/C}}{2L}t} \right).$$

This gives us a solution for I_{tr}, the *transient* current that will die out exponentially.

The particular solution will give us another term called the *steady periodic* current. It won't die out exponentially. If we run through the math, which is exactly the same as in the mechanical system, we get:

$$I_{sp}(t) = \frac{E_0 \cos{(\omega t - \alpha)}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

where

$$\alpha = \arctan\left(\frac{\omega RC}{1 - LC\omega^2}\right).$$

The quantity in the denominator of our steady periodic current is denoted by the variable Z and is called the *impedence* of the circuit. The term $\omega L - 1/(\omega C)$ is called the *reactance*.

If we use the letter Z to denote the impedence, so

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

then the amplitude of our steady periodic current is

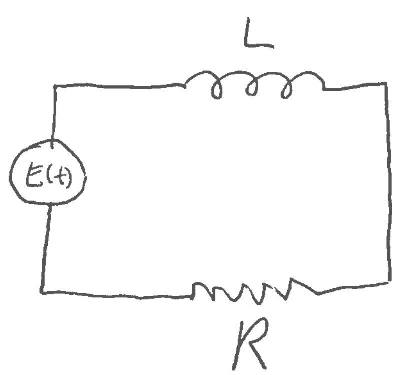
$$I_0 = \frac{E_0}{Z}.$$

If $R \neq 0$ then $Z \neq 0$, and we see that this amplitude is maximized when Z is minimized. The frequency ω that minimizes the impedence will be the frequency that makes the reactance 0. Specifically,

$$\omega_m^2 = \frac{1}{LC}.$$

This frequency is called the *resonant frequency* of the circuit.

Example - In the circuit below, suppose that $L = 2, R = 40, E(t) = 100e^{-10t}$, and $I(0) = 0$. Find the maximum current for $t \geq 0$.



More room for the example problem.

$$I'(2) + I(40) = 100e^{-10t}$$

$$I' + 20I = 50e^{-10t}$$

Using section 1.5

$$\Rightarrow \frac{d}{dt}\left(Ie^{20t}\right) = 50e^{10t}$$

$$\Rightarrow \quad Ie^{20t} = \int 50e^{10t}\,dt = 5e^{10t} + C$$

$$\Rightarrow I(t) = Ce^{-20t} + 5e^{-10t}$$

$$I(0) = C + 5 = 0 \Rightarrow C = -5$$

$$\boxed{I(t) = 5e^{-10t} - 5e^{-20t}}$$

$$I'(t) = 100e^{-20t} - 50e^{-10t} = 0 \Rightarrow 2 = e^{10t}$$

$$\Rightarrow \quad t = \frac{\ln 2}{10}$$

$$I_{max}(t) = I\left(\frac{\ln 2}{10}\right) = 5e^{-\ln 2} - 5e^{-2\ln 2}$$

$$= \frac{5}{2} - \frac{5}{4} = \boxed{\frac{5}{4}\text{ amps}}$$

Example - The parameters of an RLC circuit with input voltage $E(t)$ are:

$$R = 30\Omega,\ L = 10H,\ C = 0.02F;\ E(t) = 50\sin(2t)V.$$

Substitute

$$I_{sp}(t) = A\cos\omega t + B\sin\omega t$$

using the appropriate value of ω to find the steady periodic current in the form $I_{sp}(t) = I_0 \sin(\omega t - \delta)$.

$$\omega = 2$$

$$I_0 = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{50}{\sqrt{900 + (20-25)^2}}$$

$$= \frac{50}{\sqrt{925}} = \frac{10}{\sqrt{37}}$$

$$\alpha = \arctan\left(\frac{2(30)(.02)}{1 - (10)(.02)(2^2)}\right) = \arctan(6)$$

$$I_{sp} = \frac{10}{\sqrt{37}} \cos(2t - \arctan(6))$$

$$= \frac{10}{\sqrt{37}} \sin\left(2t - \arctan(6) + \frac{\pi}{2}\right)$$

$$= \frac{10}{\sqrt{37}} \sin\left(2t - \left(\arctan(6) - \frac{\pi}{2}\right)\right)$$

$$\text{or} \quad \frac{10}{\sqrt{37}} \sin\left(2t - \left(\frac{3\pi}{2} + \arctan(6)\right)\right)$$