# Math 2280 - Lecture 12

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Today we're going to leave the world of second-order linear ODEs for a while, and take a look at the wider world of $n$th-order linear ODEs. In particular, we'll look at how some of the ideas we talked about yesterday in the context of second-order ODEs (superposition, existence and uniqueness, linear independence, Wronskians) generalize to linear ODEs of arbitrary order. We'll also briefly discuss what to do when the linear ODE is not homogeneous, a subject to which we'll be returning in later lectures.

The assigned problems for this section are:

Section 3.2 - 1, 10, 16, 24, 31

## General Solutions of Linear ODEs

Last lecture we looked at second-order linear ODEs of the form:

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Now, it only takes a little imagination to generalize this to $n$th-order linear ODEs:

$$P_0(x)y^{(n)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x).$$

If we assume $P_0(x) \neq 0$ on $I$, our interval of interest, then we can divide both sides by $P_0(x)$ to get:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x).$$

The linear ODE has the corresponding homogeneous equation:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

As in the second-order case, for a homogeneous nth-order ODE we have that any linear combination of solutions to the ODE:

$$y = c_1 y_1 + \cdots c_k y_k$$

is also a solution. This is called the *principle of superposition*. It's proven in essentially exactly the same way as we proved it for second-order linear homogeneous ODEs.

## Existence and Uniqueness Theorems for nth-Order ODEs

Just as with first and second-order linear ODEs we have a very useful theorem regarding existence and uniqueness of solutions for $n$th-order ODEs.

**Theorem** - If $p_1, \ldots, p_n$ are continuous on an open interval $I$, and $f(x)$ is too, then the linear ODE:

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y = f(x)$$

has a unique solution satisfying the initial conditions:

$$y(a) = b_0, y'(a) = b_1, \ldots, y^{(n-1)}(a) = b_{n-1}$$

for $b_k \in \mathbb{R}$ and $a \in I$.

It's pretty much just a generalization of the existence and uniqueness theorems we've seen for first and second-order linear ODEs.

Next, we want to talk about how we can build solutions from known linearly independent solutions. Suppose we have a homogeneous nth-order linear ODE:

$$y^{(n)} + p_1 y^{(n-1)} + \cdots p_{n-1} y' + p_n y = 0,$$

and we have $n$ solutions $y_1, \ldots, y_n$. Can we get *all* solutions from solutions of the form:

$$y = c_1 y_1 + \cdots + c_n y_n?$$

Well, as with second-order linear homogeneous ODEs the answer is yes, *if* the $y_k$ are linearly independent. Again, just like with second-order linear homogeneous ODEs we can check for linear independence by using the Wronskian. Before we get any further, we should formally define what we mean by linear independence.

**Definition** - A set of functions $f_1, \ldots, f_n$ are linearly independent on an interval $I$ provided:

$$c_1 f_1 + \cdots c_n f_n = 0$$

has no solutions on $I$ except the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

Now, how do we tell if a set of $n$ functions is linearly independent? We look at the Wronskian.

$$W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Again just like the second-order case if everything is continuous on our interval of interest, then the Wronskian is either never 0 or always 0. If it's never 0 then the functions are linearly independent, if it's 0 then the functions are linearly dependent.

We don't have time to go over the proof of the statements from the above paragraph, but it's in the textbook, and it's actually pretty easy once we have the existence and uniqueness theorem.

*Example* - Use the Wronskian to prove that the functions

$$f(x) = 1 \qquad g(x) = x \qquad h(x) = x^2$$

are linearly independent on the real line.

$$f(x) = 1 \qquad g(x) = x \qquad h(x) = x^2$$

$$f'(x) = 0 \qquad g'(x) = 1 \qquad h'(x) = 2x$$

$$f''(x) = 0 \qquad g''(x) = 0 \qquad h''(x) = 2$$

$$\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

So, linearly independent on the entire real line.

## Nonhomogeneous Solutions

So far today we've just discussed homogeneous differential equations, but what happens if we have have a non-homogeneous differential equation? Good question! Suppose we do have a non-homogeneous differential equation:

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = f(x),$$

where $f(x) \neq 0$.

Now, if we have a solution, which for right now we'll call $y_{p_1}$, and another solution, which we'll call $y_{p_2}$, then if we take the difference of these two solutions $y_{p_1} - y_{p_2}$ this difference will solve the homogeneous equation:

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = 0.$$

So, the difference will be a linear combination of $n$ linearly independent solutions to the homogeneous differential equation. In other words:

$$y_{p_2} = y_{p_1} + c_1 y_1 + c_2 y_2 + \cdots c_n y_n,$$

where $y_{p_1}$ is a particular solution to the given nonhomogeneous equation, and the $y_i$ are $n$ linearly independent solutions to the attendant homogeneous equation.

OK, so what does this mean? It means that if we can find just one solution to the nonhomogeneous equation, then we've found all the other solutions if we can also solve the attendant homogeneous equation. Finding one solution can be hard, and finding the solutions of the attendant homogeneous equation can also be hard, but in many situations we'll find out it's possible and not *that* hard.

*Example*
Find the solution to the initial value problem:

$$y'' - 4y = 12$$

$$y(0) = 0, y'(0) = 10$$

Noting that a solution to the given differential equation is $y_p = -3$.

$$y'' - 4y = 0 \quad \text{has solutions}$$

$$r^2 - 4 = 0 \implies r = \pm 2. \quad \text{So,} \quad e^{2x}, e^{-2x}$$

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} - 3$$

$$y(0) = c_1 + c_2 - 3 = 0$$

$$y'(0) = 2c_1 - 2c_2 = 10 \quad \implies \quad c_1 + c_2 = 3$$

$$c_1 - c_2 = 5$$

$$\implies 2c_1 = 8$$

$$\implies c_1 = 4$$

$$c_2 = -1$$

$$\boxed{y(x) = 4e^{2x} - e^{-2x} - 3}$$