Math 2280 - Practice Final Exam

University of Utah

Spring 2013

Name: Solution Key

This is a 2 hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt.$$

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

1. **Basic Definitions** (10 points)

   Circle or state the correct answer to the questions about the following differential equation:

$$x^2y'' - \sin(x)y' + y^3 = e^{2x}$$

   (2 point) The differential equation is: Linear **Nonlinear**

   (2 points) The order of the differential equation is: 2

   For the differential equation:

$$(x^4 - x)y^{(3)} + 2xe^xy' - 3y = \sqrt{x - \cos(x)}$$

   (2 point) The differential equation is: **Linear** Nonlinear

   (2 point) The order of the differential equation is: 3

   (2 point) The corresponding homogeneous equation is:

$$(x^4 - x)y^{(3)} + 2xe^xy' - 3y = 0.$$

2. **Phase Diagrams** (15 points)

For the autonomous differential equation:

$$\frac{dx}{dt} = x^2 - 5x + 4$$

Find all critical points, draw the corresponding phase diagram, and indicate whether the critical points are stable, unstable, or semi-stable.

Solution - To find the critical points we factor the polynomial

$$x^2 - 5x + 4 = (x - 4)(x - 1).$$

The critical points are the roots of the polynomial, namely $x = 1$ and $x = 4$. The phase diagram is:

From this we can see that $x = 1$ is a stable critical point, while $x = 4$ is unstable.

3. **Ordinary Points, Regular Singular Points, and Irregular Singular Points** (15 points)

   Determine if $x = 0$ is an ordinary, regular singular, or irregular singular point in each of the following differential equations: (9 points)

   a) (5 points)

$$3x^3y'' + 2x^2y' + (1 - x^2)y = 0$$

Solution - If we divide through by $3x^3$ we get

$$P(x) = \frac{2x^2}{3x^3} = \frac{2}{3x},$$
$$Q(x) = \frac{1 - x^2}{3x^3}.$$

Neither are analytic at $x = 0$, so it's not an ordinary point. Solving for $p(x)$ and $q(x)$ we get:

$$p(x) = xP(x) = \frac{2}{3},$$
$$q(x) = x^2Q(x) = \frac{1 - x^2}{3x}.$$

The function $p(x)$ is analytic at $x = 0$ (it's a constant), but $q(x)$ is not. So, $x = 0$ is an *irregular singular point*.

   b) (5 points)

$$x^2(1 - x^2)y'' + 2xy' - 2y = 0$$

Solution - If we divide through by $x^2(1 - x^2)$ we get

$$y'' + \frac{2x}{x^2(1 - x^2)}y' - \frac{2}{x^2(1 - x^2)}y = 0.$$

The functions $P(x)$ and $Q(x)$ are:

$$P(x) = \frac{2}{x(1-x^2)},$$

$$Q(x) = \frac{2}{x^2(1-x^2)},$$

neither of which are analytic at $x = 0$. So, $x = 0$ is not an ordinary point. If we examine the functions $p(x) = xP(x)$ and $q(x) = x^2Q(x)$ we get

$$p(x) = \frac{2}{1-x^2},$$

$$q(x) = \frac{2}{1-x^2},$$

both of which (they're the same function, after all) are analytic at $x = 0$. So, $x = 0$ is a *regular singular point*.

c) (5 points)

$$xy'' + x^2y' + (e^x - 1)y = 0$$

Solution - If we divide through by x we get

$$y'' + xy' + \frac{(e^x - 1)}{x}y = 0.$$

The functions $P(x)$ and $Q(x)$ are:

$$P(x) = x,$$

$$Q(x) = \frac{e^x - 1}{x},$$

both of which are analytic at $x = 0$. We can see $Q(x)$ is analytic by noting that

$$\frac{e^x - 1}{x} = \frac{\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1}{x} = \frac{\sum_{n=1}^{\infty} \frac{x^n}{n!}}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}.$$

So, $x = 0$ is an *ordinary singular point*.

4. **Separable Ordinary Differential Equations** (20 points)

Solve the initial value problem

$$\frac{dy}{dx} = 4x^3y - y;$$

$$y(1) = -3$$

Solution - We can separate this equation as:

$$\frac{dy}{y} = (4x^3 - 1)dx.$$

Integrating both sides we get:

$$\ln y = x^4 - x + C.$$

Solving for y, and playing a bit fast and loose with the arbitrary constant, we get:

$$y = Ce^{x^4 - x}.$$

Plugging in the initial condition we get:

$$-3 = Ce^{1^4 - 1} = Ce^0 = C.$$

So,

$$y(x) = -3e^{x^4 - x}.$$

5. **Linear First-Order ODEs** (20 points)

Solve the initial value problem

$$(1+x)y' + y = \cos x;$$
$$y(0) = 1$$

We can rewrite this as

$$y' + \frac{1}{1+x}y = \frac{\cos x}{1+x}.$$

So, $P(x) = \frac{1}{1+x}$, and our integrating factor is

$$e^{\int P(x)dx} = e^{\int \frac{1}{1+x}dx} = e^{\ln 1+x} = 1+x.$$

So, we can rewrite our differential equation as:

$$\frac{d}{dx}((1+x)y) = \cos x.$$

Integrating both sides we get:

$$(1+x)y = \sin x + C,$$

and solving for y we get:

$$y(x) = \frac{\sin x + C}{1+x}.$$

Plugging in our initial condition $y(0) = 1$ we get

$$1 = \frac{\sin 0 + C}{1 + 0} = C.$$

So, our solution is:

$$y(x) = \frac{\sin x + 1}{1 + x}.$$

6. **Nonhomogeneous Linear Differential Equations** (25 points)

Find the general solution to the differential equation

$$y'' - y' - 6y = 2x + e^{-2x}.$$

Solution - First, we solve the homogeneous equation:

$$y'' - y' - 6y = 0.$$

The characteristic equation is $r^2 - r - 6 = (r-3)(r+2)$. So, the roots of the characteristic equation are $r = 3, -2$, and the homogeneous solution is:

$$y_h = c_1 e^{3x} + c_2 e^{-2x}.$$

Now, we need a particular solution to the nonhomogeneous solution. The equation $2x + e^{-2x}$ is the sum of a first-order polynomial and an exponential, so we'll "guess" our solution will be too:

$$y_p = A + Bx + Ce^{-2x}.$$

However, we've got a problem here. The function e^{-2x} already shows up in our homogeneous solution. So, we need to replace Ce^{-2x} in our guess with Cxe^{-2x}. If we do this and plug our guess into our ODE we get:

$$y_p'' - y_p' - 6y_p$$

$$= 4Cxe^{-2x} - 4Ce^{-2x} - B + 2Cxe^{-2x} - Ce^{-2x} - 6A - 6Bx - 6Cxe^{-2x}$$

$$= (-6A - B) - 6Bx - 5Ce^{-2x}.$$

Equating this with the right-hand side of our differential equation we get:

$$-6A - B = 0,$$

$$-6B = 2,$$

$$-5C = 1.$$

So, $A = \frac{1}{18}, B = -\frac{1}{3}, C = -\frac{1}{5}$. Using these values, our general solution is:

$$y(x) = c_1 e^{3x} + c_2 e^{-2x} + \frac{1}{18} - \frac{1}{3}x - \frac{1}{5}xe^{-2x}.$$

7. **Systems of Differential Equations** (30 points)

Find the general solution to the system of differential equations

$$\begin{array}{rcrcrcr} x_1' & = & 5x_1 & + & x_2 & + & 3x_3 \\ x_2' & = & x_1 & + & 7x_2 & + & x_3 \\ x_3' & = & 3x_1 & + & x_2 & + & 5x_3 \end{array}$$

Hint: $\lambda = 2$ is an eigenvalue of the coefficient matrix, and all eigenvalues are real.

Solution - The corresponding matrix equation is:

$$\mathbf{x}' = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{pmatrix} \mathbf{x}.$$

The eigenvalues of the matrix are:

$$\begin{vmatrix} 5-\lambda & 1 & 3 \\ 1 & 7-\lambda & 1 \\ 3 & 1 & 5-\lambda \end{vmatrix} = -\lambda^3 + 17\lambda^2 - 84\lambda + 108$$

$$= -(\lambda - 2)(\lambda - 6)(\lambda - 9).$$

So, the eigenvalues are $\lambda = 2, 6, 9$.

The corresponding eigenvectors will be:

For $\lambda = 2$:

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

is an eigenvector for $\lambda = 2$.

For $\lambda = 6$:

$$\begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = 6$.

For $\lambda = 9$:

$$\begin{pmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda = 9$.

So, the general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{6t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t}.$$

8. **Systems of Differential Equations with Repeated Eigenvalues** (25 points)

Find the general solution to the system of differential equations:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix} \mathbf{x}.$$

Solution - The eigenvalues of the matrix are:

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = (1-\lambda)(9-\lambda) + 16 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

So, $\lambda = 5$ is the only eigenvalue. To get a second solution, we'll need to find a generalized eigenvector. So, we'll need a length 2 chain:

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1,$$

$$(A - \lambda I)\mathbf{v}_1 = \mathbf{0}.$$

So, $(A - \lambda I)^2\mathbf{v}_1 = \mathbf{0}$. Calculating $(A - \lambda I)^2$ we get:

$$(A - \lambda I)^2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, *any* vector $\mathbf{v}_2$ that is not already an eigenvector of A will work. Let's make it easy on ourselves and pick

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From this we get

$$\mathbf{v}_1 = (A - \lambda I)\mathbf{v}_2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

So, our solutions will be:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{5t},$$

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{5t}.$$

So, our general solution is:

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = c_1 \begin{pmatrix} -4 \\ 4 \end{pmatrix} e^{5t} + c_2 \left[\begin{pmatrix} -4 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{5t}.$$

9. **Laplace Transforms** (10 points)

   Using the definition of the Laplace transform, calculate the Laplace transform of the function

   $$f(t) = e^{3t+1},$$

   and state its domain.

   Solution - The Laplace transform of $f(t)$ will be:

   $$F(s) = \int_0^\infty e^{3t+1}e^{-st}dt = e\int_0^\infty e^{(3-s)t}dt = \left.\frac{e^{(3-s)t+1}}{3-s}\right|_0^\infty = \frac{e}{s-3}.$$

   Here we've assumed $s > 3$, which is the domain of $F(s)$.

10. **Laplace Transforms and Differential Equations** (25 points)

Find a particular solution to the initial value problem:

$$x'' + 4x = \delta(t) + \delta(t - \pi);$$

$$x(0) = x'(0) = 0.$$

Solution - Taking the Laplace transform of out terms we get:

$$\mathcal{L}(x'') = s^2 X(s) - sx(0) - x'(0) = s^2 X(s),$$

$$\mathcal{L}(x) = X(s),$$

$$\mathcal{L}(\delta(t) + \delta(t - \pi)) = 1 + e^{-\pi s}.$$

Plugging these in we get:

$$(s^2 + 4)X(s) = 1 + e^{-\pi s},$$

and so

$$X(s) = \frac{1 + e^{-\pi s}}{s^2 + 4}.$$

We have

$$\mathcal{L}^{-1}\left(\frac{1}{s^2 + 4}\right) = \frac{1}{2}\sin 2t.$$

Using this and the relation

$$u(t - a)f(t - a) = e^{-as}F(s)$$

we get

$$x(t) = \frac{1}{2}\sin 2t + \frac{u(t - \pi)}{2}\sin 2(t - \pi).$$

11. **Power Series** (30 points)

Solve the following second-order ODE using power series methods:

$$y'' + x^2 y' + 2xy = 0.$$

Solution - $x = 0$ is an ordinary point, and we have

$$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}.$$

Plugging these into our ODE we get:

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^{n+1} + 2 \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

The first non-zero, or at least not automatically zero, power of x is x^0. For this term $n = 2$ in the frist series, while the other series don't enter into it. So,

$$2(2-1)c_2 x^0 = 0x^0 \Rightarrow 2c_2 = 0 \Rightarrow c_2 = 0.$$

So, $c_2 = 0$. On the other hand, c_0 and c_1 are "arbitrary" (they would be determined by the initial conditions) and for higher order powers we get the recurrence relation:

$$(n+3)(n+2)c_{n+3} + (n+2)c_n = 0.$$

As $n \geq 0$ we can divide by $(n+2)$ and get the relation:

$$c_{n+3} = -\frac{c_n}{n+3}.$$

This gives us the terms:

$$c_0 = c_0,$$

$$c_3 = -\frac{c_0}{3},$$

$$c_6 = -\frac{c_3}{6} = \frac{c_0}{6 \times 3},$$

$$c_9 = -\frac{c_6}{9} = -\frac{c_0}{9 \times 6 \times 3},$$

and in general

$$c_{3n} = \frac{c_0(-1)^n}{3^n n!}.$$

As for the c_{3n+1} terms we have:

$$c_1 = c_1,$$

$$c_4 = -\frac{c_1}{4},$$

$$c_7 = -\frac{c_4}{7} = \frac{c_1}{7},$$

$$c_{10} = -\frac{c_7}{10} = -\frac{c_1}{10 \times 7 \times 4},$$

and in general

$$c_{3n+1} = \frac{c_1(-1)^n}{1 \times 4 \times 7 \times \cdots \times (3n+1)}.$$

As $c_2 = 0$ all terms of the form c_{3n+2} will be zero. So, our solution will be:

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{3^n n!} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{1 \times 4 \times 7 \times \cdots \times (3n+1)}.$$

12. **Fourier Series** (25 points)

The values of the periodic function $f(t)$ in one full period are given. Find the function's Fourier series.

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \\ 0 & t = \{-2, 0\} \end{cases}$$

Extra Credit (5 points) - Use this solution and what you know about Fourier series to deduce the famous Leibniz formula for π.

Solution - We first note that $f(t)$ is odd, so all the a_n terms in the Fourier series will be zero. The period here is $4 = 2L$, so the b_n Fourier coefficients are:

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t) \sin\frac{n\pi t}{2} dt$$

(noting $f(t)$ is odd, so $f(t)\sin\frac{n\pi t}{2}$ is even)

$$= \int_0^2 f(t) \sin\frac{n\pi t}{2} dt$$

$$= \int_0^2 \sin\frac{n\pi t}{2} dt = -\frac{2}{n\pi}\cos\frac{n\pi t}{2}\bigg|_0^2 = -\frac{2}{n\pi}((-1)^n - 1)$$

$$= \begin{cases} 0 & n \; even \\ \frac{4}{n\pi} & n \; odd \end{cases}$$

So, our Fourier series is

$$f(t) \sim \frac{4}{\pi}\sum_{n \; odd} \frac{\sin\left(\frac{n\pi t}{2}\right)}{n}.$$

If we plug in $t = 1$ we get:

$$f(1) = 1 = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \cdots\right)$$

$$= \frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots),$$

and so

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots)$$

which is the famous Leibniz formula for π!