Math 2280 - Practice Final Exam

University of Utah

Spring 2013

Name: ______________________________

This is a 2 hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^nX(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

1. **Basic Definitions** (10 points)

   Circle or state the correct answer to the questions about the following differential equation:

$$x^2y'' - \sin(x)y' + y^3 = e^{2x}$$

   (2 point) The differential equation is: Linear Nonlinear

   (2 points) The order of the differential equation is:

   For the differential equation:

$$(x^4 - x)y^{(3)} + 2xe^x y' - 3y = \sqrt{x - \cos(x)}$$

   (2 point) The differential equation is: Linear Nonlinear

   (2 point) The order of the differential equation is:

   (2 point) The corresponding homogeneous equation is:

2. **Phase Diagrams** (15 points)

   For the autonomous differential equation:

$$\frac{dx}{dt} = x^2 - 5x + 4$$

   Find all critical points, draw the corresponding phase diagram, and indicate whether the critical points are stable, unstable, or semi-stable.

3. **Ordinary Points, Regular Singular Points, and Irregular Singular Points** (15 points)

   Determine if $x = 0$ is an ordinary, regular singular, or irregular singular point in each of the following differential equations: (9 points)

   a) (5 points)

$$3x^3y'' + 2x^2y' + (1 - x^2)y = 0$$

   b) (5 points)

$$x^2(1 - x^2)y'' + 2xy' - 2y = 0$$

c) (5 points)

$$xy'' + x^2y' + (e^x - 1)y = 0$$

4. **Separable Ordinary Differential Equations** (20 points)
Solve the initial value problem

$$\frac{dy}{dx} = 4x^3y - y;$$
$$y(1) = -3$$

5. **Linear First-Order ODEs** (20 points)

   Solve the initial value problem

$$(1+x)y' + y = \cos x;$$

$$y(0) = 1$$

6. **Nonhomogeneous Linear Differential Equations** (25 points)
   Find the general solution to the differential equation

$$y'' - y' - 6y = 2x + e^{-2x}.$$

More room for this problem, if you need it.

7. **Systems of Differential Equations** (30 points)

Find the general solution to the system of differential equations

$$\begin{array}{rcrcrcr} x_1' & = & 5x_1 & + & x_2 & + & 3x_3 \\ x_2' & = & x_1 & + & 7x_2 & + & x_3 \\ x_3' & = & 3x_1 & + & x_2 & + & 5x_3 \end{array}$$

Hint: $\lambda = 2$ is an eigenvalue of the coefficient matrix, and all eigenvalues are real.

More room for this problem, if you need it.

8. **Systems of Differential Equations with Repeated Eigenvalues** (25 points)

   Find the general solution to the system of differential equations:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix} \mathbf{x}.$$

More room for this problem, if you need it.

9. **Laplace Transforms** (10 points)

Using the definition of the Laplace transform, calculate the Laplace transform of the function

$$f(t) = e^{3t+1},$$

and state its domain.

10. **Laplace Transforms and Differential Equations** (25 points)

Find a particular solution to the initial value problem:

$$x'' + 4x = \delta(t) + \delta(t - \pi);$$

$$x(0) = x'(0) = 0.$$

More room for the problem, if you need it.

11. **Power Series** (30 points)

Solve the following second-order ODE using power series methods:

$$y'' + x^2 y' + 2xy = 0.$$

More room for the problem, if you need it.

Even *more* room for the problem, if you need it.

12. **Fourier Series** (25 points)

The values of the periodic function $f(t)$ in one full period are given. Find the function's Fourier series.

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \\ 0 & t = \{-2, 0\} \end{cases}$$

Extra Credit (2 points) - Use this solution and what you know about Fourier series to deduce the famous Leibniz formula for π.

More room for the problem, if you need it.