Math 2280 - Final Exam

University of Utah

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This is a 2 hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction. There are 250 possible points on this exam.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right).$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt.$$

1. **Basic Definitions** (10 points)

   Circle or state the correct answer to the questions about the following differential equation:

$$\sqrt{x}y^{(5)} - x^2(y')^2 + e^x y = \cos x$$

   (2 point) The differential equation is: Linear **Nonlinear**

   (2 points) The order of the differential equation is: 5

   For the differential equation:

$$x^2y'' + xy' - y = \sin\left(e^{x^2+5x+2}\right)$$

   (2 point) The differential equation is: **Linear** Nonlinear

   (2 point) The order of the differential equation is: 2

   (2 point) The corresponding homogeneous equation is:

$$x^2y'' + xy' - y = 0.$$

2. **Phase Diagrams** (15 points)

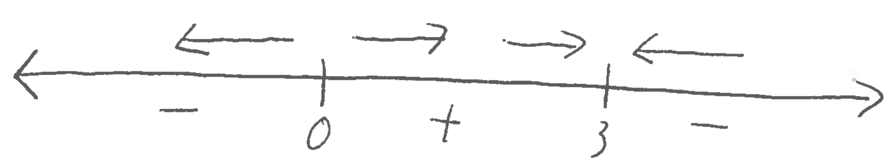
For the autonomous differential equation:

$$\frac{dx}{dt} = 3x - x^2$$

Find all critical points, draw the corresponding phase diagram, and indicate whether the critical points are stable, unstable, or semi-stable.

Solution - The roots of the function $3x - x^2 = x(3-x)$ are $x = 0, 3$. So, the critical points are $x = 0$ and $x = 3$.

The phase diagram is draw below:



We can see from the phase diagram above that $x = 0$ is an unstable critical point, while $x = 3$ is a stable critical point.

3. **Ordinary Points, Regular Singular Points, and Irregular Singular Points** (15 points)

Determine if $x = 0$ is an ordinary, regular singular, or irregular singular point in each of the following differential equations:

a) (5 points)

$$x(1+x)y'' + 2y' + 3xy = 0$$

Solution - The function

$$P(x) = \frac{2}{x(1+x)}$$

is singular at $x = 0$, while the functions

$$p(x) = xP(x) = \frac{2}{(1+x)},$$

and

$$q(x) = x^2Q(x) = \frac{3x^3}{x(1+x)} = \frac{3x^2}{(1+x)},$$

are non-singular at $x = 0$. So, $x = 0$ is a regular singular point.

b) (5 points)

$$x^3y'' + 2x^2y' + 7y = 0$$

Solution - The function

$$Q(x) = \tfrac{7}{x^3}$$

is singular at $x = 0$, as is the function

$$q(x) = x^2Q(x) = \tfrac{7}{x}.$$

So, $x = 0$ is an irregular singular point.

c) (5 points)

$$x(1-x)(1+x)y'' + x^2y' + x^3y = 0$$

Solution - The functions

$$P(x) = \frac{x^2}{x(1-x)(1+x)} = \frac{x}{(1-x)(1+x)},$$

and

$$Q(x) = \frac{x^3}{x(1-x)(1+x)} = \frac{x^2}{(1-x)(1+x)}$$

are both nonsingular at $x = 0$. So, $x = 0$ is an ordinary point.

4. **Indicial Equations** (15 points)

What are the roots of the indicial equation for differential equation:

$$x^2y'' + 3(\sin x)y' + e^x y = 0.$$

Will the method of Frobenius be guaranteed to yield two linearly independent solutions? Could it possibly yield two linearly independent solutions? Why or why not?[1]

Solution - We have:

$$p_0 = \lim_{x\to 0} p(x) = \lim_{x\to 0} \frac{3x \sin x}{x^2} = \lim_{x\to 0} \frac{3 \sin x}{x} = 3,$$

and

$$q_0 = \lim_{x\to 0} q(x) = \lim_{x\to 0} \frac{x^2 e^x}{x^2} = \lim_{x\to 0} e^x = 1.$$

So, the indicial equation is:

$$r(r-1) + p_0 r + q_0 = r(r-1) + 3r + 1 = r^2 + 2r + 1 = (r+1)^2.$$

The indicial equation has a single root, of multiplicity 2, at $r = -1$.

As there is only one root, there is only one possible Fourier series solution, a solution of the form:

$$y(x) = x^{-1} \sum_{n=0}^{\infty} c_n x^n.$$

[1]Note - You aren't expected to find the solutions here.

5. **Undetermined Coefficients** (10 points)

What is the form of the particular solution to the following differential equation:

$$y^{(3)} + y'' + y' + y = x^2 e^{-5x} \sin(3x),$$

using the method of undetermined coefficients?[2]

Solution -

$$y_p = (Ax^2 + Bx + C)e^{-5x} \sin(3x) + (Dx^2 + Ex + F)e^{-5x} \cos(3x).$$

[2]You don't have to solve the differential equation, nor do you have to find the coefficients! You just have to give the form of the particular solution dictated by the method of undetermined coefficients. So, for example, if the differential equation were $y' + 3y = \sin x$ the particular solution would be of the form $y_p = A\cos x + B\sin x$. I'm just asking for that, I'm not asking you to find A and B.

6. **Nonhomogeneous Linear Differential Equations with Constant Coefficients** (30 points)

   Find the general solution to the differential equation

$$y'' + 7y' + 12y = x + e^{-4x}.$$

Solution - The characteristic equation is:

$$r^2 + 7r + 12 = (r+4)(r+3).$$

The roots of the characteristic equation are $r = -4, -3$, and so the homogeneous solution is:

$$y_h = c_1 e^{-3x} + c_2 e^{-4x}.$$

The method of undetermined coefficients tells us our particular solution should be of the form:

$$y_p = Ax + B + Ce^{-4x}.$$

However, e^{-4x} is not linearly independent of our homogeneous solution. So, we need to modify our "guess":

$$y_p = Ax + B + Cxe^{-4x}.$$

Differentiating this we get:

$$y_p = Ax + B + Cxe^{-4x},$$

$$y_p' = A - 4Cxe^{-4x} + Ce^{-4x},$$

$$y_p'' = 16Cxe^{-4x} - 8Ce^{-4x}.$$

Plugging these into our differential equation we get:

$$12Ax + (7A + 12B) - Ce^{-4x} = x + e^{-4x}.$$

Solving for our coefficients we get:

$$A = \frac{1}{12}$$
$$B = -\frac{7}{12}A = -\frac{7}{144}$$
$$C = -1.$$

So, our particular solution is:

$$y_p = \frac{1}{12}x - \frac{7}{144} - xe^{-4x},$$

and our complete solution is:

$$y = c_1e^{-3x} + c_2e^{-4x} + \frac{1}{12}x - \frac{7}{144} - xe^{-4x}.$$

7. **Systems of Differential Equations** (35 points)

Find the general solution to the system of differential equations

$$\mathbf{x}' = \begin{pmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix} \mathbf{x}$$

Solution - The matrix

$$\begin{pmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

has the eigenvalue equation

$$\begin{vmatrix} -2-\lambda & -9 & 0 \\ 1 & 4-\lambda & 0 \\ 1 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)((-2-\lambda)(4-\lambda)+9)$$
$$= (1-\lambda)(\lambda^2 - 2\lambda + 1) = (1-\lambda)(\lambda-1)^2 = -(\lambda-1)^3.$$

So, there is only one root to the eigenvalue equation, namely $\lambda = 1$.

The eigenvector equation:

$$\begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

has the two linearly independent eigenvectors:

$$\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We need three linearly independent solutions, so we want to construct a generalized eigenvector. We note

$$(A - \lambda I)^2 = \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, any vector that isn't an eigenvector will do. If we choose

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

we get

$$(A - \lambda I)\mathbf{v} = \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}.$$

So, our general solution will be:

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t + c_3 \left(\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) e^t.$$

8. **Laplace Transforms and Convolutions** (15 points)

Using the definition of convolution, calculate the convolution of the functions:

$$f(t) = t,$$
$$g(t) = e^t.$$

What is the Laplace transform of $f(t) * g(t)$? In other words, what is $\mathcal{L}(f(t) * g(t))$?[3]

Solution - The convolution will be:

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau = e^t \int_0^t \tau e^{-\tau} d\tau$$
$$= -e^t(\tau e^{-\tau} + e^{-\tau})|_0^t = e^t - 1 - t.$$

To calculate $\mathcal{L}(f(t) * g(t))$ we can use the relation:

$$\mathcal{L}(f(t) * g(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t)).$$

As

$$\mathcal{L}(e^t) = \frac{1}{s-1},$$

and

$$\mathcal{L}(t) = \frac{1}{s^2},$$

[3] If you try to answer this second question using the formal definition of the Laplace transform, you're doing it the hard way.

we have

$$\mathcal{L}(t * e^t) = \frac{1}{s^2(s-1)}.$$

If we were to do this the harder way we'd get:

$$\mathcal{L}(f(t) * g(t)) = \mathcal{L}(e^t - 1 - t) = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} = \frac{1}{s^2(s-1)}.$$

OK, I suppose it's not really *that* much harder.

9. **Power Series** (30 points)

Solve the following second-order ODE using power series methods:

$$(x^2 + 2)y'' + 4xy' + 2y = 0.$$

Solution - We represent our solution $y(x)$ as a power series:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

The derivatives of $y(x)$, represented as power series, will be:

$$y'(x) = \sum_{n=0}^{\infty} c_n n x^{n-1},$$

and

$$y''(x) = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}.$$

Plugging these into our differential equation we get:

$$\sum_{n=0}^{\infty} c_n n(n-1)x^n + \sum_{n=0}^{\infty} 2c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} 4nc_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0.$$

Noting that we can rewrite the second summation as

$$\sum_{n=0}^{\infty} 2c_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} 2c_{n+2}(n+2)(n+1)x^n,^4$$

[4] The $n = 0$ and $n = 1$ terms on the left are both 0, so that's why it's OK to say both sides start at $n = 0$.

we get the recurrence relation:

$$c_n n(n-1) + 2c_{n+2}(n+2)(n+1) + 4nc_n + 2c_n = 0.$$

Simplifying this we get:

$$c_{n+2} = -\frac{n^2+3n+2}{2(n+2)(n+1)}c_n = -\frac{c_n}{2}.$$

So, the constants c_0, c_1 are arbitrary, and we get:

$$c_{2n} = \frac{(-1)^n c_0}{2^n},$$

and

$$c_{2n+1} = \frac{(-1)^n c_1}{2^n}.$$

Our general solution is:

$$y(x) = c_0 \sum_{n=0}^{\infty} \left(\frac{-x^2}{2}\right)^n + c_1 x \sum_{n=0}^{\infty} \left(\frac{-x^2}{2}\right)^n.$$[5]

Using our geometric series formula we can rewrite this as:

$$y(x) = \frac{2c_0 + 2c_1 x}{2 + x^2},$$

and the interval of our solution is $-\sqrt{2} < x < \sqrt{2}$, which is as expected.

[5]If you just got this far on the final, it would be full credit.

10. **Fourier Series** (25 points)

The values of the periodic function $f(t)$ in one full period are given. Find the function's Fourier series.

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \\ 0 & t = \{-2, 0\} \end{cases}$$

Extra Credit (5 points) - Use this solution and what you know about Fourier series to deduce the famous Leibniz formula for π.

Solution - We first note that $f(t)$ is odd, so all the a_n terms in the Fourier series will be zero. The period here is $4 = 2L$, so the b_n Fourier coefficients are:

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{n\pi t}{2}dt$$

(noting $f(t)$ is odd, so $f(t)\sin\frac{n\pi t}{2}$ is even)

$$= \int_0^2 f(t)\sin\frac{n\pi t}{2}dt$$

$$= \int_0^2 \sin\frac{n\pi t}{2}dt = -\frac{2}{n\pi}\cos\frac{n\pi t}{2}\bigg|_0^2 = -\frac{2}{n\pi}((-1)^n - 1)$$

$$= \begin{cases} 0 & n\ even \\ \frac{4}{n\pi} & n\ odd \end{cases}$$

So, our Fourier series is

$$f(t) \sim \frac{4}{\pi}\sum_{n\ odd}\frac{\sin\left(\frac{n\pi t}{2}\right)}{n}.$$

If we plug in $t = 1$ we get:

$$f(1) = 1 = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \cdots\right)$$

$$= \frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots),$$

and so,

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots)$$

which is the famous Leibniz formula for π!

11. **Fixed Endpoint Problem** (20 points)

For the fixed endpoint problem:

$$X'' + \lambda X = 0,$$

$$X(0) = X(2) = 0;$$

what are the possible eigenvalues λ_n, and the corresponding eigenfunctions X_n?

Solution - We'll examine the three cases: $\lambda < 0, \lambda = 0, \lambda > 0$ individually.

For $\lambda < 0$ the solutions will be of the form:

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

If we plug in $X(0) = 0$ we get:

$$A + B = 0.$$

If we plug in $X(2) = 0$ we get:

$$Ae^{2\sqrt{-\lambda}} + Be^{-2\sqrt{-\lambda}} = 0.$$

From the first equation we know $A = -B$, and so plugging this into the second equation we get:

$$A(e^{2\sqrt{-\lambda}} - e^{-2\sqrt{-\lambda}}) = 0.$$

If $A \neq 0$ we must have:

$$e^{2\sqrt{-\lambda}} = e^{-2\sqrt{-\lambda}},$$

which implies

$$e^{4\sqrt{-\lambda}} = 1.$$

This is only possible if $4\sqrt{-\lambda} = 0$, which would only be possible if $\lambda = 0$, which it is not. So, there are no non-trivial solutions for $\lambda < 0$.

For $\lambda = 0$ the solution must be of the form:

$$X(x) = Ax + B.$$

As $X(0) = 0$ this means $B = 0$. If $X(2) = 0$ we must have $2A = 0$, and so $A = 0$. Therefore, for $\lambda = 0$, there are no non-trivial solutions.

For $\lambda > 0$ our solutions will be of the form:

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

If we plug in $X(0) = 0$ we get $A = 0$. If we plug in $X(2) = 0$ we get:

$$B\sin 2\sqrt{\lambda} = 0.$$

If $B \neq 0$ we must have $\sin 2\sqrt{\lambda} = 0$. We know $\sin x = 0$ only when $x = n\pi$, and so we must have

$$\lambda_n = \frac{n^2\pi^2}{4}.$$

These are the possible eigenvalues. The corresponding eigenfunctions are:

$$X_n(x) = \sin\frac{n\pi x}{2}.$$

12. **The Heat Equation** (30 points)

Solve the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

with the boundary values:

$$u(0,t) = u(2,t) = 0,$$

$$u(x,0) = \left\{ \begin{array}{ll} 1 & 0 < x < 2 \\ 0 & x = \{0,2\} \end{array} \right.$$

Note - The solutions to the last two problems might be useful to you here.

Solution - The solutions to the previous two problems are *very* useful to us here.

I'll go through the derivation of the solution in detail. What we're looking for is a function, $u(x,t)$, that satisfies the partial differential equation above, and satisfies the given boundary values.

We "guess" that our solution will be separable. More specifically, we guess it will be of the form:

$$u(x,t) = X(x)T(t),$$

Plugging this into our partial differential equation we get:

$$X(x)T'(t) = X''(x)T(t).$$

We can rewrite this to get:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

If we modify x but keep t constant, neither side above can change. If we modify t but keep x constant, neither side above can change. The only way this is possible is if the equality above is equal to a constant, which we'll denote $-\lambda$.

So,

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

We can rewrite these as the linear differential equations:

$$X'' + \lambda X = 0,$$

and

$$T' + \lambda T = 0.$$

The first differential equation is the endpoint value problem:

$$X'' + \lambda X = 0,$$

with the endpoint values $u(0,t) = X(0) = 0$, and $u(2,t) = X(2) = 0$. Hmmm.... does that look familiar? It should, because it's the fixed endpoint problem from earlier in the exam. Taking the solution from that problem we get that the possible values of λ are:

$$\lambda_n = \frac{n^2\pi^2}{4},$$

with corresponding eigenfunctions:

$$X_n = \sin\left(\frac{n\pi x}{2}\right).$$

The solution to the differential equation:

$$T' + \lambda T = 0,$$

is a constant multiple of $e^{-\lambda t}$. So, for λ_n we have:

$$T_n = e^{-\frac{n^2\pi^2 t}{4}}.$$

The corresponding product of functions will be

$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{n^2\pi^2 t}{4}}.$$

Our final solution will be a linear combination of these functions:

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{n^2\pi^2 t}{4}}.$$

The final question is, what are the c_n coefficients? Well, we want to choose them so that:

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right) = \left\{ \begin{array}{ll} 1 & 0 < x < 2 \\ 0 & x = \{0,2\} \end{array} \right.$$

Look familiar? Well, again, it should. This is the Fourier series problem we did earlier. The coefficients calculated there were:

$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

So, our final solution is:

$$u(x,t) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{n^2\pi^2 t}{4}}.$$

This can also be written as:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi x}{2}\right) e^{-\frac{(2n+1)^2\pi^2 t}{4}}.$$

It's up to you which of the two forms above you like. Either one is fine. It's really an aesthetic decision.

And that's it! Thank you all so much for a great semester. I hope you enjoyed and benefited from the class. Have a good summer, and good luck and best wishes in all you do.

-Dylan