Math 2280 - Exam 4

University of Utah

Spring 2013

Name: *Solutions*

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

1. (15 points) *The Laplace Transform*

   Calculate the Laplace transform of the function

$$f(t) = e^{at}$$

   using the definition of the Laplace transform.

   Solution - Using the definition of the Laplace transform we want to calculate the integral:

$$\int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt$$

$$= \left. \frac{e^{(a-s)t}}{a-s} \right|_0^\infty = -\frac{1}{a-s} = \frac{1}{s-a}.$$

   We must assume here that $s > a$.

2. (25 points) *Laplace Transforms and Initial Value Problems*

Use Laplace transforms to solve the initial value problem

$$x'' - x' - 2x = 0$$

$$x(0) = 0 \quad x'(0) = 2.$$

Solution - We use the derivative formulas:

$$\mathcal{L}(x'') = s^2 X(s) - sx(0) - x'(0),$$

$$\mathcal{L}(x') = sX(s) - x(0).$$

If we take the Laplace transform of the differential equation above, applying the derivative formulas, and plugging in the given values for $x(0)$ and $x'(0)$ we get:

$$s^2 X(s) - 2 - sX(s) - 2X(s) = 0.$$

If we solve this for $X(s)$ we get:

$$X(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s-2)(s+1)}.$$

We want the the partial fraction decomposition of the rational function above, which will be:

$$\frac{2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} = \frac{(A+B)s + (A-2B)}{(s-2)(s+1)}.$$

So, we have the two linear equations:

$$A + B = 0,$$

$$A - 2B = 2.$$

Solving these for A and B gives us $A = \frac{2}{3}$ and $B = -\frac{2}{3}$. So, plugging these values into our partial fraction decomposition we get:

$$X(s) = \frac{2}{3}\left(\frac{1}{s-2}\right) - \frac{2}{3}\left(\frac{1}{s+1}\right).$$

Taking the inverse Laplace transform of $X(s)$ gives us our solution:

$$x(t) = \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t}.$$

3. (15 points) *Convolutions*

Calculate the convolution, $f(t) * g(t)$, of the following functions

$$f(t) = e^{at}, \qquad g(t) = e^{bt}.\ (a \neq b)$$

Solution - This is just a matter of calculating the integral:

$$e^{at} * e^{bt} = \int_0^t e^{a\tau} e^{b(t-\tau)} d\tau.$$

Taking this integral gives us:

$$\int_0^t e^{a\tau} e^{b(t-\tau)} d\tau = e^{bt} \int_0^t e^{(a-b)\tau} d\tau = \left. \frac{e^{bt}}{a-b} e^{(a-b)\tau} \right|_0^t$$

$$= \frac{e^{at} - e^{bt}}{a-b}.$$

4. (25 points) *Delta Functions*

Solve the initial value problem

$$x'' + 2x' + x = t + \delta(t)$$

$$x(0) = 0 \quad x'(0) = 1.$$

Solution - If we take the Laplace transform of both sides of the initial value problem above we get:

$$s^2 X(s) - 1 + 2sX(s) + X(s) = \frac{1}{s^2} + 1.$$

Solving this for $X(s)$ gives us:

$$X(s) = \frac{2s^2 + 1}{s^2(s+1)^2}.$$

Calculating the partial fraction decomposition of this rational function gives us:

$$\frac{2s^2 + 1}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

$$= \frac{(A+C)s^3 + (2A + B + C + D)s^2 + (A + 2B)s + B}{s^2(s+1)^2}.$$

This gives us four linear equations:

$$A + C = 0,$$

$$2A + B + C + D = 2,$$

$$A + 2B = 0,$$

$$B = 1.$$

These are quite easily solved to get $A = -2, B = 1, C = 2, D = 3$. So, our partial fraction decomposition is:

$$X(s) = \frac{-2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{3}{(s+1)^2}.$$

The inverse Laplace transform of $X(s)$, our solution $x(t)$, is:

$$x(t) = (t - 2) + 2e^{-t} + 3te^{-t}.$$

5. (10 points) *The Ratio Test*

   Use the ratio test to determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n$$

   Solution - We want to use the ratio test to calculate the radius of convergence, which means we want to calculate the limit:

$$\lim_{n \to \infty} \left(\frac{\frac{n+1}{3^n}}{\frac{n+2}{3^{n+1}}} \right) = \lim_{n \to \infty} 3 \left(\frac{n+1}{n+2} \right) = 3.$$

   So, the power series has radius of convergence 3, and converges for $|x| < 3$.