Math 2280 - Practice Exam 4

University of Utah

Spring 2013

Name: *Solution Key*

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (15 points) *The Laplace Transform*

   Calculate the Laplace transform of the function

$$f(t) = t^2$$

   using the definition of the Laplace transform.

   Solution - Using the definition of the Laplace transform we get:

$$\mathcal{L}(t^2) = \int_0^\infty t^2 e^{-st} dt = -t^2 \frac{e^{-st}}{s}|_0^\infty + \frac{2}{s}\int_0^\infty te^{-st} dt$$

$$= \frac{1}{s}\int_0^\infty 2te^{-st} dt = -2t\frac{e^{-st}}{s}|_0^\infty + \frac{2}{s^2}\int_0^\infty e^{-st} dt = 0 - \frac{2e^{-st}}{s^3}|_0^\infty = \frac{2}{s^3}.$$

   Where here we've just used integration by parts multiple times. Note we need to assume $s > 0$.

2. (25 points) *Laplace Transforms and Initial Value Problems*

Use Laplace transforms to solve the initial value problem

$$x'' - 6x' + 8x = 2$$

$$x(0) = x'(0) = 0.$$

Solution - Using the formula for taking the Laplace transform of a derivative, we get that the Laplace transform of the left side of the differential equation is:

$$(s^2 X(s) - sx(0) - x'(0)) - 6(sX(s) - x(0)) + 8X(s).$$

Plugging in $x(0) = x'(0) = 0$ we get

$$s^2 X(s) - 6sX(s) + 8X(s) = (s^2 - 6s + 8)X(s) = (s - 4)(s - 2)X(s).$$

The Laplace transform of the right side is

$$\mathcal{L}(2) = \frac{2}{s}.$$

So, taking the Laplace transform of both sides of the equality gives us:

$$(s - 4)(s - 2)X(s) = \frac{2}{s},$$

and so

$$X(s) = \frac{2}{s(s - 4)(s - 2)}.$$

We need to do a partial fraction decomposition on the right hand side. If we do this we get

$$\frac{2}{s(s-4)(s-2)} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s-2}$$

$$= \frac{(A+B+C)s^2 - (6A+2B+4C)s + 8A}{s(s-4)(s-2)}.$$

So, we have the three equations

$$A + B + C = 0,$$

$$6A + 2B + 4C = 0,$$

$$8A = 2.$$

Solving these three linear equations for A, B, and C we get $A = \frac{1}{4}, B = \frac{1}{4}, C = -\frac{1}{2}$.

So, the Laplace transform is

$$X(s) = \frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{4}\left(\frac{1}{s-4}\right) - \frac{1}{2}\left(\frac{1}{s-2}\right).$$

Taking the inverse Laplace transform gives us

$$x(t) = \frac{1}{4} + \frac{1}{4}e^{4t} - \frac{1}{2}e^{2t},$$

which is the solution to the initial value problem.

3. (15 points) *Convolutions*

Calculate the convolution, $f(t) * g(t)$, of the following functions

$$f(t) = t, \qquad g(t) = e^{at}.$$

The convolution $f(t) * g(t)$ is

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)dt = \int_0^t \tau e^{a(t-\tau)}d\tau.$$

Calculating this integral we get:

$$\int_0^t \tau e^{at}e^{-a\tau}d\tau = e^{at}\int_0^t \tau e^{-a\tau}d\tau$$

$$= e^{at}\left(-\frac{\tau e^{-a\tau}}{a}|_0^t + \frac{1}{a}\int_0^t e^{-a\tau}d\tau\right) = -\frac{t}{a} - e^{at}\left(\frac{1}{a^2}e^{-a\tau}|_0^t\right)$$

$$= -\frac{t}{a} - \frac{1}{a^2} + \frac{e^{at}}{a^2} = \frac{e^{at} - at - 1}{a^2}.$$

4. (25 points) *Delta Functions*

Solve the initial value problem

$$x'' + 2x' + x = \delta(t) - \delta(t-2)$$
$$x(0) = x'(0) = 2.$$

Solution - Taking the Laplace transform of both sides of the equation we get

$$(s^2X(s) - 2s - 2) + 2(sX(s) - 2) + X(s) = 1 - e^{-2s}.$$

Simplifying this we get

$$(s^2 + 2s + 1)X(s) = 2s + 7 - e^{-2s},$$

and solving this for $X(s)$ we get

$$X(s) = \frac{2s + 7 - e^{-2s}}{(s+1)^2} = 2\frac{s+1}{(s+1)^2} + 5\frac{1}{(s+1)^2} - \frac{e^{-2s}}{(s+1)^2}.$$

Taking the inverse Laplace transform of this we get:

$$x(t) = 2e^{-t} + 5te^{-t} - u(t-2)(t-2)e^{-(t-2)}.$$

5. (10 points) *The Ratio Test*

   Use the ratio test to determine the radius of convergence of the geometric series

$$\sum_{n=0}^{\infty} x^n$$

Solution - The coefficients in the power series above are all 1. So, $c_n = 1$ for all n. Plugging this into the ratio test we get

$$\lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} 1 = 1.$$

So, the radius of convergence is $\rho = 1$, and the geometric series converges for $|x| < 1$.