Math 2280 - Exam 3

University of Utah

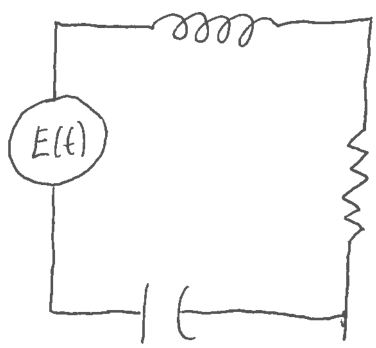
Spring 2013

Name: Solutions

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (15 points) *An RLC Circuit*

For the RLC circuit pictured below:



Calculate $I(t)$ for the values:

$$L = 10H \quad R = 20\Omega \quad C = 0.02F$$

$$E(t) = 50 \sin 3t \text{ V}$$

More room, if necessary, for Problem 1.

$$10Q'' + 20Q' + \frac{1}{.02}Q = 50\sin(3t)$$

$$Q'' + 2Q' + 5Q = 5\sin(3t)$$

$$\frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2} = -1 \pm 2i$$

$$\Rightarrow Q_h(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$

$$Q_p(t) = A\sin(3t) + B\cos(3t)$$

$$Q_p'(t) = 3A\cos(3t) - 3B\sin(3t)$$

$$Q_p''(t) = -9A\sin(3t) - 9B\cos(3t)$$

$$(-4A - 6B)\sin(3t) + (-4B + 6A)\cos(3t) = 5\sin(3t)$$

$$\begin{aligned} -4A - 6B &= 5 \\ 6A - 4B &= 0 \end{aligned} \Rightarrow \begin{aligned} -12A - 18B &= 15 \\ 12A - 8B &= 0 \end{aligned} \Rightarrow -26B = 15$$

$$B = -\frac{15}{26} \qquad \frac{3}{2}A = B \Rightarrow A = \frac{2}{3}B = \frac{2}{3}\left(-\frac{15}{26}\right) = -\frac{15}{39}$$

$$\Rightarrow Q_p = -\frac{15}{39}\sin(3t) - \frac{15}{26}\cos(3t) + c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$

$$\Rightarrow \boxed{I(t) = -\frac{15}{13}\cos(3t) + \frac{45}{26}\sin(3t) + \tilde{c}_1 e^{-t}\cos(2t) + \tilde{c}_2 e^{-t}\sin(2t)}$$

2. (15 points) *An Endpoint Problem*

The eigenvalues for this problem are all nonnegative. First, determine whether $\lambda = 0$ is an eigenvalue; then find the positive eigenvalues and associated eigenfunctions.

$$y'' + \lambda y = 0;$$

$$y'(-\pi) = 0 \quad y'(\pi) = 0.$$

$\lambda = 0$

$y(x) = Ax + B$

$y'(x) = A$

$y'(-\pi) = y'(\pi) = 0 \Rightarrow A = 0$

So, $y(x) = B$ and

$\lambda = 0$ is an eigenvalue

with eigenfunction

$y(x) = 1$.

$\lambda > 0$

$y(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$

$\cancel{y''(x}$

$y'(x) = -\sqrt{\lambda} A \sin(\sqrt{\lambda} x) + \sqrt{\lambda} B \cos(\sqrt{\lambda} x)$

$\cancel{y''(x)}$

$y'(-\pi) = \sqrt{\lambda} A \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda} B \cos(\sqrt{\lambda}\pi) = 0$

$y'(\pi) = -\sqrt{\lambda} A \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda} B \cos(\sqrt{\lambda}\pi) = 0$

$\Rightarrow 2\sqrt{\lambda} B \cos(\sqrt{\lambda}\pi) = 0$

True if $B = 0$ or if

$\cos(\sqrt{\lambda}\pi) = 0$. Now, $\cos(t) = 0$ for $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

$\Rightarrow \sqrt{\lambda} = \dfrac{2n+1}{2} \qquad n \in \mathbb{Z}$

$\lambda_n = \left(\dfrac{2n+1}{2}\right)^2$

Note: $A = 0$ in this case.

More room, if necessary, for Problem 2.

If $B = 0$ then

$\sqrt{\lambda} A \sin(\sqrt{\lambda}\pi) = 0$

$-\sqrt{\lambda} A \sin(\sqrt{\lambda}\pi) = 0$

$\Rightarrow A = 0$ <u>or</u>

$\sin(\sqrt{\lambda}\pi) = 0$

true if $\sqrt{\lambda} = 0, 1, 2, \ldots$

So, $\sqrt{\lambda} = n$ and $\lambda = n^2$.

So, the eigenvalues are:

$\lambda = \left(\frac{n}{2}\right)^2 \quad n \in \mathbb{Z}, n > 0$

with eigenfunctions

~~cos(√~~

~~cos((2n+1~~

$$\boxed{\begin{array}{ll} \cos\left(\frac{n}{2}x\right) & \text{for } n \text{ odd} \\ \sin\left(\frac{n}{2}x\right) & \text{for } n \text{ even.} \end{array}}$$

3. (10 points) *Converting to First-Order Systems*

Transform the given differential equation into an equivalent system of first-order differential equations:

$$x^{(4)} + 6x'' - 3x' + x = \cos 3t$$

$x = x_1$

$x_1' = x_2$

$x_2' = x_3$

$x_3' = x_4$

$x_4' = -6x_3 + 3x_2 - x_1 + \cos(3t)$

So, the system is:

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= -6x_3 + 3x_2 - x_1 + \cos(3t) \end{aligned}}$$

4. (25 points) *Systems of First-Order ODEs*

Find the general solution to the system of ODEs:

$$\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5 \\ -2t \end{pmatrix}.$$

Homogeneous solution:

$$\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$$

So, eigenvalues are $\lambda = 4, -1$

eigenvectors

$\lambda=4$ $\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ works

$\lambda=-1$ $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works.

So,

$$x_h = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

More room, if necessary, for Problem 4.

Method of undetermined coefficients:

$$\vec{x}_p = \vec{a}t + \vec{b}$$
$$x_p' = \vec{a}$$
$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{pmatrix} + \begin{pmatrix} 5 \\ -2t \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} (2a_1 + 3a_2)t + (2b_1 + 3b_2) + 5 \\ (2a_1 + a_2)t + (2b_1 + b_2) - 2t \end{pmatrix}$$

$$2a_1 + 3a_2 = 0$$
$$2b_1 + 3b_2 + 5 = a_1$$
$$2a_1 + a_2 - 2 = 0$$
$$2b_1 + b_2 = a_2$$

$$2(2b_1 + 3b_2 + 5) + 3(2b_1 + b_2) = 0$$
$$2(2b_1 + 3b_2 + 5) + (2b_1 + b_2) - 2 = 0$$

$$10b_1 + 9b_2 = -10 \qquad 60b_1 + \cancel{90}\,54b_2 = -60$$
$$6b_1 + 7b_2 = -8 \quad \Rightarrow \quad -60b_1 - 70b_2 = 80$$

$$-16b_2 = 20$$
$$b_2 = -\frac{5}{4}$$
$$6b_1 = -8 + \frac{35}{4}$$
$$6b_1 = \cancel{-\frac{67}{4}}\ \frac{3}{4}$$
$$\cancel{b_1 = -\frac{67}{24}} \quad b_1 = \frac{1}{8}$$

$$a_1 = 2\left(\frac{1}{8}\right) + 3\left(-\frac{5}{4}\right) + 5$$
$$= \frac{1}{4} - \frac{15}{4} + \frac{20}{4} = \frac{3}{2}$$

$$a_2 = 2\left(\frac{1}{8}\right) - \frac{5}{4} = \frac{1}{4} - \frac{5}{4} = -1$$

$$\vec{x}_p = \begin{pmatrix} \frac{3}{2} \\ -1 \end{pmatrix} t + \begin{pmatrix} \frac{1}{8} \\ -\frac{5}{4} \end{pmatrix}$$

$$\boxed{\vec{x} = \begin{pmatrix} \frac{3}{2} \\ -1 \end{pmatrix} t + \begin{pmatrix} \frac{1}{8} \\ -\frac{5}{4} \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}}$$

$$\cancel{a_1 = 2\left(-\frac{67}{24}\right)}$$

5. (20 points) *Multiple Eigenvalues*[1]

Find the general solution to the system of ODEs:

$$\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \mathbf{x}.$$

$$\begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = (\lambda-5)^2$$

$\lambda = 5, 5$ repeated eigen value

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$ works. Only one!

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \; !$$

So, try $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

So, solution is: $\boxed{\vec{x} = c_1 \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{5t} + c_2 \left[\begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{5t}}$

[1] This is a hint.

More room, if necessary, for Problem 5.

6. (15 points) *Matrix Exponentials*

Calculate the matrix exponential e^A for the matrix:

$$\begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= B} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 3 & 0 \end{pmatrix}}_{= C}$$

$$C^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{pmatrix} \qquad C^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$BC = CB$ as B is diagonal

$$e^B = \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix} \qquad e^C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 3 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow e^A = e^B e^C = \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ \frac{17}{2} & 3 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} e^2 & 0 & 0 \\ 3e & e & 0 \\ \frac{17}{2}e & 3e & e \end{pmatrix}}$$