Math 2280 - Practice Exam 3

University of Utah

Spring 2013

Name: Solutions

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (25 points) *An Endpoint Problem*

   The eigenvalues for this problem are all nonnegative. First, determine whether $\lambda = 0$ is an eigenvalue; then find the positive eigenvalues and associated eigenfunctions.

$$y'' + \lambda y = 0;$$

$$y'(0) = 0 \quad y'(\pi) = 0.$$

$\lambda = 0$

$y'' = 0 \Rightarrow y(x) = Ax + B$

$y'(x) = A$

$y'(0) = A = 0 \Rightarrow A = 0$

$y'(\pi) = 0.$

So, $y(x) = B$ works for any B. λ is an eigenvalue with eigenfunction $y(x) = 1$.

$\lambda > 0$

$y'' + \lambda y = 0 \quad r^2 + \lambda = 0 \quad r = \pm\sqrt{-\lambda}$

$y(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$y'(x) = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}\, x) + \sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}\, x)$

$y'(0) = \sqrt{\lambda}\, c_2 = 0 \Rightarrow c_2 = 0$ as $\sqrt{\lambda} \neq 0$

More room, if necessary, for Problem 1.

So, $y'(x) = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}\, x)$

$y'(\pi) = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}\,\pi) = 0$

$\Rightarrow \sin(\sqrt{\lambda}\,\pi) = 0$ if $c_1 \neq 0$

$\sin(t) = 0$ if $t = n\pi$ where $n \in \mathbb{Z}$.

So, $\sqrt{\lambda} = n \Rightarrow \lambda = n^2$

for $n = 1, 2, 3, \ldots$

Eigen functions $y_n(x) = \cos(nx)$

So, eigen values $\lambda = n^2$ for $n = 0, 1, 2, 3, \ldots$

w/ eigen functions $y_n = \cos(nx)$.

2. (10 points) *Converting to First-Order Systems*

Transform the given differential equation into an equivalent system of first-order differential equations:

$$t^3x^{(3)} - 2t^2x^{(2)} + 3tx' + 5x = \ln t$$

$x = x_1$

$x_2 = x_1'$

$x_3 = x_2'$

$\cancel{x_4 = x_3' = x^{(3)}}$

$t^3 x^{(3)} - 2t^2 x^2 + 3t + 5x = \ln t$

$x^{(3)} = \frac{2}{t} x^{(2)} - \frac{3}{t^2} x' - \frac{5}{t^3} x + \frac{\ln t}{t^3}$

$\Rightarrow \quad \cancel{x_4 =}$

$$\boxed{\begin{aligned} x_3' &= \frac{2}{t} x_3 - \frac{3}{t^2} x_2 - \frac{5}{t^3} x_1 + \frac{\ln t}{t^3} \\ x_2' &= x_3 \\ x_1' &= x_2 \end{aligned}}$$

3. (30 points) *Systems of First-Order ODEs*

Find the general solution to the system of ODEs:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}.$$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2-1) - (-\lambda-1) + (1+\lambda)$$

$$= -\lambda(\lambda^2-1) + 2(\lambda+1)$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= (\lambda+1)(-\lambda^2+\lambda+2)$$

$$= -(\lambda+1)(\lambda-2)(\lambda+1) = -(\lambda+1)^2(\lambda-2)$$

Roots $\lambda = -1, -1, 2$

Eigenvectors:

$\lambda = -1$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

work and are linearly independent.

$\lambda = 2$ $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ works.

More room, if necessary, for Problem 3.

So, the general solution will be:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}}$$

4. (20 points) *Multiple Eigenvalues*[1]

Find the general solution to the system of ODEs:

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} \mathbf{x}.$$

Find the characteristic equation:

$$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda)+1$$

$$= \lambda^2 - 8\lambda + 16 = (\lambda-4)^2$$

So, $\lambda = 4$ is an eigenvalue of multiplicity 2.

Eigenvector $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works and there are no other linearly independent eigenvectors

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

[1]This is a hint.

More room, if necessary, for Problem 4.

So, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ works

$$\vec{v}_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and the general solution is:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) e^{4t}}$$

5. (15 points) *Matrix Exponentials*

Calculate the matrix exponential e^A for the matrix:

$$\begin{pmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 10 & 21 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 10 & 21 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Nilpotent!

More room, if necessary, for Problem 5.

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$$

$A^k = 0$ for $k \geq 4$

So,

$$e^A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 10 & 21 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 & 2 & 8 & \frac{33}{2} \\ 0 & 1 & 5 & \frac{15}{2} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$