Math 2280 - Exam 2

University of Utah

Spring 2013

Name: Solutions

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (25 points) *Population Models*

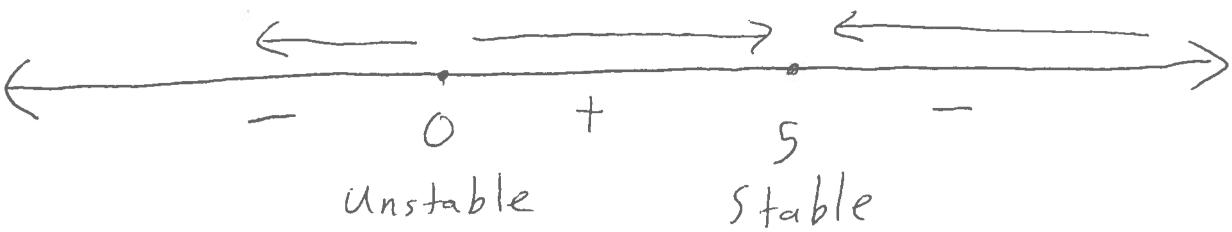
For the population model

$$\frac{dP}{dt} = 3P(5 - P)$$

what are the equilibrium populations, and for each equilibrium population determine if it is a stable or unstable. Draw the corresponding phase diagram. Then, use separation of variables to solve the population model exactly with the initial population $P(0) = 8$.

$3P(5-P) = 0$ if $P = 0$ or $P = 5$.

So, equilibrium populations are $P = 0$ or $P = 5$.



$$\frac{dP}{dt} = 3P(5-P)$$

$$\Rightarrow \frac{dP}{P(5-P)} = 3dt$$

More room for problem 1, if you need it.

$$\left(\frac{A}{P} + \frac{B}{5-P}\right) dP = 3\,dt$$

$$\frac{A(5-P) + BP}{P(5-P)} = \frac{1}{P(5-P)} \qquad P(B-A) + 5A = 1$$

$$\Rightarrow A = \frac{1}{5}$$

$$B = \frac{1}{5}$$

$$\frac{1}{5} \int \left(\frac{1}{P} + \frac{1}{5-P}\right) dP = \int 3\,dt$$

$$\Rightarrow \ln P - \ln(5-P) = 15t + C \Rightarrow \ln\left(\frac{P}{5-P}\right) = 15t + C$$

$$\Rightarrow \frac{P}{5-P} = Ce^{15t} \Rightarrow P(1 + Ce^{15t}) = 5Ce^{15t}$$

$$\Rightarrow P(t) = \frac{5Ce^{15t}}{1 + Ce^{15t}} \qquad P(0) = \frac{5C}{1+C} = 8 \Rightarrow 5C = 8 + 8C$$

$$\Rightarrow -3C = 8 \Rightarrow C = -\frac{8}{3}$$

$$P(t) = \frac{-\frac{40}{3} e^{15t}}{1 - \frac{8}{3} e^{15t}} = \frac{40e^{15t}}{8e^{15t} - 3} = \boxed{\frac{40}{8 - 3e^{-15t}}}$$

2. (20 points) *Euler's Method*

Use Euler's method with a step size $h = 1$ to calculate an approximate value for $y(3)$, where $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = -3x^2y,$$

$$y(0) = 3.$$

$y_0 = 3$ at $x = 0$

$y_1 = 3 + 1(-3(0^2)3) = 3$ at $x = 1$

$y_2 = 3 + 1(-3(1^2)(3)) = -6$ at $x = 2$

$y_3 = -6 + 1(-3(2^2)(-6)) = -6 + 72 = \boxed{66}$ at $x = 3$

More room for problem 2, if you need it.

3. (20 points) *Higher Order Linear Homogeneous Differential Equations with Constant Coefficients*

Find the general solution to the differential equation:

$$y^{(3)} + y'' - y' - y = 0.$$

Characteristic Equation:

$$r^3 + r^2 - r - 1 = (r-1)(r^2+2r+1)$$

$$= (r-1)(r+1)^2$$

So, the general solution is:

$$\boxed{y(x) = c_1 e^{x} + c_2 e^{-x} + c_3 x e^{-x}}$$

4. (15 points) *Wronskians*

Use the Wronskian to prove that the functions

$$f(x) = e^x \qquad g(x) = \cos x \qquad h(x) = \sin x$$

are linearly independent on the real line $\mathbb{R}$.

$$\begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} = e^x(\sin^2 x + \cos^2 x) - e^x(-\cos x \sin x + \cos x \sin x) + e^x(\cos^2 x + \sin^2 x)$$

$$= 2e^x \neq 0 \quad \text{on } \mathbb{R}.$$

5. (20 points) *Nonhomogeneous Linear Equations*

Find the solution to the linear ODE:

$$y'' + 3y' + 2y = e^{-x},$$

with initial conditions

$$y(0) = 0 \qquad\qquad y'(0) = 3.$$

Characteristic equation

$r^2 + 3r + 2 = (r+2)(r+1)$

$y_h = c_1 e^{-2x} + c_2 e^{-x}$

$y_p = Ae^{-x}$ doesn't work, so we need

$y_p = Axe^{-x}$

$y_p' = -Axe^{-x} + Ae^{-x}$

$y_p'' = Axe^{-x} - 2Ae^{-x}$

$y_p'' + 3y_p' + 2y_p = Axe^{-x} - 2Ae^{-x} - 3Axe^{-x} + 3Ae^{-x} + 2Axe^{-x}$

$= Ae^{-x} = e^{-x}$

So, $A = 1$.

More room for problem 5, if you need it.

$$y(x) = c_1 e^{-2x} + c_2 e^{-x} + xe^{-x}$$

$$y'(x) = -2c_1 e^{-2x} - c_2 e^{-x} - xe^{-x} + e^{-x}$$

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = -2c_1 - c_2 + 1 = 3$$

So,

$$c_1 + c_2 = 0$$
$$-2c_1 - c_2 = 2 \quad \Rightarrow \quad c_2 = 2, \quad c_1 = -2$$

Ergo,

$$\boxed{y(x) = -2e^{-2x} + 2e^{-x} + xe^{-x}}$$