Math 2280 - Practice Exam 2

University of Utah

Spring 2013

Name: Solutions

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

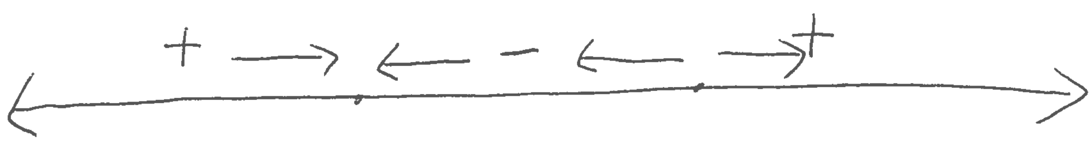
1. (25 points) *Population Models*

   For the population model

$$\frac{dP}{dt} = 3P(P-5)$$

   what are the equilibrium populations, and for each equilibrium population determine if it is a stable or unstable. Draw the corresponding phase diagram. Then, use separation of variables to solve the population model exactly with the initial population $P(0) = 2$.

$$\frac{dP}{dt} = 0 \quad \text{if} \quad 3P(P-5) = 0 \Rightarrow \boxed{P = 0 \text{ or } P = 5}$$



$P = 0$ Stable $\qquad P = 5$ Unstable

$$\frac{dP}{P(P-5)} = 3\,dt \qquad \frac{A}{P} + \frac{B}{P-5} = \frac{1}{P(P-5)}$$

$$\Rightarrow \frac{1}{5}\int\left(\frac{1}{P-5} - \frac{1}{P}\right)dP = \int 3\,dt \qquad (A+B)P - 5A = 1$$

$$\Rightarrow A + B = 0 \Rightarrow A = -B$$

$$\ln\left(\frac{P-5}{P}\right) = 15t + C \qquad -5A = 1 \Rightarrow A = -\frac{1}{5}$$

$$B = \frac{1}{5}$$

More room for problem 1, if you need it.

$$\frac{P-5}{P} = Ce^{15t}$$

$$P-5 = PCe^{15t}$$

$$P(1-Ce^{15t}) = 5$$

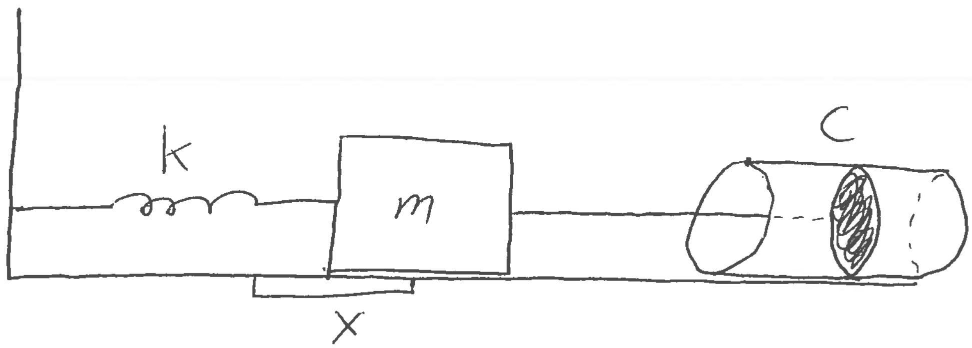
$$P(t) = \frac{5}{1-Ce^{15t}} \qquad P(0) = \frac{5}{1-C} = 2$$

$$5 = 2-2C \Rightarrow C = -\frac{3}{2}$$

$$P(t) = \frac{5}{1+\frac{3}{2}e^{15t}} = \boxed{\frac{10}{2+3e^{15t}}}$$

Note: $\lim_{t\to\infty} P(t) = 0$ as expected.

2. (20 points) *Mechanical Systems*



For the mass-spring-dashpot system drawn[1] above, find the equation that describes its motion with the parameters:

$$m = 3;$$

$$c = 30;$$

$$k = 63;$$

and initial conditions:

$$x_0 = 2; \qquad v_0 = 2.$$

Is the system overdamed, underdamped, or critically damped?

[1]Poorly.

More room for problem 2, if you need it.

$$3x'' + 30x' + 63x = 0$$

$$\Rightarrow x'' + 10x' + 21x = 0$$

Characteristic equation:

$r^2 + 10r + 21$ roots are $\frac{-10 \pm \sqrt{10^2 - 4(1)(21)}}{2}$

$$= \frac{-10 \pm \sqrt{16}}{2} = \frac{-10 \pm 4}{2} = -7, -3$$

a.k.a. $r^2 + 10r + 21$ factors as $(r+7)(r+3)$

$\boxed{\text{Overdamped}}$

$$x(t) = c_1 e^{-7t} + c_2 e^{-3t}$$

$$x'(t) = -7c_1 e^{-7t} - 3c_2 e^{-3t}$$

$x(0) = c_1 + c_2 = 2$ $\quad$ $x'(0) = -7c_1 - 3c_2 = 2$

$c_2 = 2 - c_1$ $\quad$ $-7c_1 - 3(2 - c_1) = 2$

$$-4c_1 = 8 \Rightarrow c_1 = -2$$

$$c_2 = 4$$

~~$x(t) = 14e^{-7t} - 12$~~ $\quad$ $\boxed{x(t) = 4e^{-3t} - 2e^{-7t}}$

3. (20 points) *Higher Order Linear Homogeneous Differential Equations with Constant Coefficients*

Find the general solution to the differential equation:

$$y^{(4)} + 2y^{(3)} + y'' - 12y' + 8 = 0.$$

Hint: The polynomial $x^4 + 2x^3 + x^2 - 12x + 8$ has $x = 1$ as a root of multiplicity 2.

Characteristic equation:

$$r^4 + 2r^3 + r^2 - 12r + 8 = (r-1)^2(r^2+4r+8)$$

$$\frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2} = -2 \pm 2i$$

So, general solution is:

$$\boxed{y(x) = c_1 e^x + c_2 x e^x + c_3 e^{-2x}\cos(2x) + c_4 e^{-2x}\sin(2x)}$$

4. (15 points) *Wronskians*

Use the Wronskian to prove that the functions

$$f(x) = e^{x} \qquad g(x) = e^{2x} \qquad h(x) = e^{3x}$$

are linearly independent on the real line $\mathbb{R}$.

$$W(x) = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} e^{6x}$$

$$= (18+3+4-12-9-2)e^{6x}$$

$$= \boxed{2e^{6x}}$$

So, $W(x) \neq 0$, and therefore the functions are linearly independent.

5. (20 points) *Nonhomogeneous Linear Equations*

Find the solution to the linear ODE:

$$y'' - y' - 2y = 3x + 4,$$

with initial conditions

$$y(0) = \frac{7}{4} \qquad y'(0) = \frac{3}{2}.$$

$y_p = Ax + B$

$y_p' = A \qquad y_p'' - y_p' - 2y_p$

$y_p'' = 0 \qquad = 0 - A - 2Ax - 2B = 3x + 4$

$\Rightarrow -2Ax - (A + 2B) = 3x + 4$

$-2A = 3 \Rightarrow A = -\frac{3}{2}$

$-A - 2B = 4 \Rightarrow \frac{3}{2} - 2B = 4 \Rightarrow -2B = \frac{5}{2}$

$\Rightarrow B = -\frac{5}{4}$

$y_p = -\frac{3}{2}x - \frac{5}{4}$

More room for problem 5, if you need it.

$$r^2 - r - 2 = (r-2)(r+1)$$

So, $y_c = c_1 e^{2x} + c_2 e^{-x}$

$$y(x) = c_1 e^{2x} + c_2 e^{-x} - \frac{3}{2}x - \frac{5}{4}$$

$$y'(x) = 2c_1 e^{2x} - c_2 e^{-x} - \frac{3}{2}$$

$$y(0) = c_1 + c_2 - \frac{5}{4} = \frac{7}{4} \Rightarrow c_1 + c_2 = 3$$

$$y'(0) = 2c_1 - c_2 - \frac{3}{2} = \frac{3}{2} \Rightarrow 2c_1 - c_2 = 3$$

$$\Rightarrow 3c_1 = 6 \Rightarrow c_1 = 2, \quad c_2 = 1$$

So,

$$\boxed{y(x) = 2e^{2x} + e^{-x} - \frac{3}{2}x - \frac{5}{4}}$$