Math 2280 - Exam 1

University of Utah

Spring 2013

Name: Solutions

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (15 Points) *Differential Equation Basics*

(a) (5 points) What is the order of the differential equation given below?[1]

$$y'x^3 + (y'')^2 + y^{(3)}y^2 \sin(x) = 14x^5 + 7x^2 - e^{-x^2}$$

3rd order

(b) (5 points) Is the differential equation given below linear?

$$xy'y + 2xy^2 = \cos e^x$$

No. The $y'y$ term makes it nonlinear.

(c) (5 points) On what intervals are we guaranteed a unique solution exists for the differential equation below?

$$y' + \frac{y}{x} = \frac{x+3}{x^2-1}$$

$P(x) = \frac{1}{x}$ is continuous for $x \neq 0$

$Q(x) = \frac{x+3}{x^2-1} = \frac{x+3}{(x+1)(x-1)}$ is continuous for $x \neq \pm 1$

So, the intervals are:

$(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$

[1]Extra credit - Solve this differential equation! Just kidding. Do not attempt to solve it.

2. (25 points) *Separable Equations*

Find the general solution to the differential equation given below.

$$\frac{dy}{dx} - 3\sqrt{xy} = 0$$

$$\frac{dy}{dx} = 3\sqrt{xy} \Rightarrow \int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x}\,dx$$

$$2\sqrt{y} = 2x^{3/2} + C$$

$$\Rightarrow \sqrt{y} = x^{3/2} + C$$

$$\boxed{y = (x^{3/2} + C)^2}$$

Check:

$$\frac{dy}{dx} = 2(x^{3/2} + C)\frac{3}{2}\sqrt{x}$$

$$= 3(x^{3/2} + C)\sqrt{x}$$

$$= 3\sqrt{x}\sqrt{(x^{3/2} + C)^2} = 3\sqrt{xy} \quad \checkmark$$

3. (30 points) *Exact Equations*

Find the solution to the initial value problem given below.[2]

$$\frac{dy}{dx} = -\frac{\cos x + \ln y}{\frac{x}{y} + e^y}$$

with initial condition $y(0) = 5$.

Rewrite as:

$$\left(\frac{x}{y} + e^y\right)dy + (\cos x + \ln y)\,dx = 0$$

$$\frac{\partial}{\partial y}(\cos x + \ln y) = \frac{1}{y} \qquad \frac{\partial}{\partial x}\left(\frac{x}{y} + e^y\right) = \frac{1}{y}$$

So, exact.

$$\int \left(\frac{x}{y} + e^y\right)dy = x\ln y + e^y + g(x)$$

$$\frac{\partial}{\partial x}\left(x\ln y + e^y + g(x)\right) = \ln y + g'(x) = \cos x + \ln y$$

$$\Rightarrow g'(x) = \cos(x) \Rightarrow g(x) = \sin(x) + C$$

So, general solution is:

$$x\ln y + e^y + \sin x = C$$

$x = 0 \quad y = 5$ means

$$e^5 = C$$

So, $\boxed{x\ln y + e^y + \sin x = e^5}$

[2]The title of this problem is a hint.

4. (30 points) *First-Order Linear Equations*

Find a solution to the initial value problem given below, and give the interval upon which you know the solution is unique.

$$(1+x)y' + y = \cos x \qquad y(0) = 1.$$

$$\frac{dy}{dx} + \left(\frac{1}{1+x}\right) y = \frac{\cos x}{1+x}$$

$$e^{\int \frac{1}{1+x} dx} = e^{\ln(1+x)} = (1+x)$$

So,

$$(1+x)\frac{dy}{dx} + y = \cos x$$

$$\Rightarrow \frac{d}{dx}\left[(1+x)y\right] = \cos x$$

$$\Rightarrow (1+x)y = \sin x + C \Rightarrow y(x) = \frac{\sin x + C}{1+x}$$

$$y(0) = \frac{\sin(0) + C}{1+0} = C = 1. \text{ So, } C = 1$$

$$\boxed{y(x) = \frac{\sin x + 1}{x+1}}$$

$\frac{1}{1+x}$ and $\frac{\cos x}{1+x}$ are continuous for $x \neq -1$. So, unique on the interval $\boxed{(-1, \infty)}$.