Math 2280 - Practice Exam 1

University of Utah

Spring 2013

Name: Solution

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (15 Points) *Differential Equation Basics*

(a) (5 points) What is the order of the differential equation given below?

$$(y'')^2 + 2xy' - \sin(x)y^3 = 14x^2 + e^x$$

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(b) (5 points) Is the differential equation given below linear?

$$y^{(3)} + 2x^2y'' - 4\sin(x)y' + e^x y = x^4 + x^2e^x$$

$\boxed{\text{Yes}}$

(c) (5 points) On what intervals are we guaranteed a unique solution exists for the differential equation below?

$$y' + \ln(x)y = \frac{x+3}{x^2-1}$$

$\ln(x)$ is defined for $x > 0$, and $\frac{x+3}{x^2-1}$ is defined for $x \neq \pm 1$. So, we're guaranteed a unique solution exists for an initial value (a, b) if $a \in (0, 1)$ or $a > 1$.

2. (25 points) *Separable Equations*

Find the general solution to the differential equation given below.

$$\frac{dy}{dx} + 2xy^2 = 0$$

$$\frac{dy}{dx} = -2xy^2$$

$$\Rightarrow \frac{dy}{y^2} = -2x\,dx$$

$$\Rightarrow \int \frac{dy}{y^2} = \int -2x\,dx$$

$$\Rightarrow -\frac{1}{y} = -x^2 + C$$

$$\Rightarrow \boxed{y = \frac{1}{x^2 + C}}$$

3. (25 points) *Substitution Methods*

Find the general solution to the differential equation given below.

$$y' = y + y^3$$

$y' - y = y^3$ Bernoulli equation.

$v = y^{-2}$ $\quad \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$

$\Rightarrow y' = -\frac{y^3}{2} v'$

$y' - y = y^3 \Rightarrow \frac{y'}{y^3} - \frac{1}{y^2} = 1$

$\Rightarrow \frac{y'}{y^3} = -\frac{v'}{2} \quad -\frac{1}{y^2} = -v$

$\Rightarrow -\frac{v'}{2} - v = 1 \Rightarrow v' + 2v = -2$

$e^{2x}v' + 2e^{2x}v = -2e^{2x}$

$\Rightarrow \frac{d}{dx}(e^{2x}v) = -2e^{2x} \Rightarrow e^{2x}v = -e^{2x} + C$

$v(x) = \frac{C}{e^{+2x}} - 1$

$\Rightarrow \frac{1}{y^2} = \frac{C}{e^{2x}} - 1$

$\boxed{y^2 = \frac{1}{Ce^{-2x} - 1}}$

4. (10 points) *Existence and Uniqueness*

For what initial conditions $y(a) = b$ does the following differential equation have a unique solution in an interval around a?

$$\frac{dy}{dx} = \sqrt{x - y}$$

$f(x,y) = \sqrt{x-y}$ is defined for $x \geq y$.

$\frac{\partial f}{\partial y} = -\frac{1}{2\sqrt{x-y}}$ is defined for $x > y$.

Also, both continuous for $x > y$.

So, for ~~$y(a)$~~ $y(a) = b$ if $a > b$ a unique solution exists in an interval around a.

5. (25 points) *First-Order Linear Equations*

Find a solution to the initial value problem given below, and explain, if it's true, why the solution is unique.

$$xy' - y = x \qquad\qquad y(1) = 7.$$

$$y' - \frac{y}{x} = 1 \qquad e^{\int -\frac{dx}{x}} = e^{-\ln x + C} = \frac{C}{x}$$

$$\cancel{\frac{d}{dx}(} \qquad \frac{y'}{x} - \frac{y}{x^2} = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x}$$

$$\Rightarrow \frac{y}{x} = \ln(x) + C$$

$$y(x) = x\ln(x) + Cx$$

$$y(1) = 1\ln(1) + C(1) = 7$$

$$\boxed{y(x) = x\ln(x) + 7x}$$

$$y' - \left(\frac{1}{x}\right)y = 1$$

Both $-\frac{1}{x}$ and 1 are continuous at $x=1$, so the solution is unique.