

# Math 2280 - Assignment 3

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**Section 2.1** - 1, 8, 11, 16, 29

**Section 2.2** - 1, 10, 21, 23, 24

**Section 2.3** - 1, 2, 4, 10, 24

## Section 2.1 - Population Models

2.1.1 Separate variables and use partial fractions to solve the initial value problem:

$$\frac{dx}{dt} = x - x^2 \qquad x(0) = 2.$$

More space, if necessary, for problem 2.1.1.

**2.1.8** Separate variables and use partial fractions to solve the initial value problem:

$$\frac{dx}{dt} = 7x(x - 13) \qquad x(0) = 17.$$

More space, if necessary, for problem 2.1.8.

**2.1.11** Suppose that when a certain lake is stocked with fish, the birth and death rates  $\beta$  and  $\delta$  are both inversely proportional to  $\sqrt{P}$ .

**(a)** Show that

$$P(t) = \left( \frac{1}{2}kt + \sqrt{P_0} \right)^2.$$

**(b)** If  $P_0 = 100$  and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

More space, if necessary, for problem 2.1.11.

**2.1.16** Consider a rabbit population  $P(t)$  satisfying the logistic equation  $dP/dt = aP - bP^2$ . If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time  $t = 0$ , how many months does it take for  $P(t)$  to reach 95% of the limiting population  $M$ ?



More space, if necessary, for problem 2.1.16.

**2.1.29** During the period from 1790 to 1930 the U.S. population  $P(t)$  ( $t$  in years) grew from 3.9 million to 123.2 million. Throughout this period,  $P(t)$  remained close to the solution of the initial value problem

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2, \quad P(0) = 3.9.$$

- (a) What 1930 population does this logistic equation predict?
- (b) What limiting population does it predict?
- (c) Has this logistic equation continued since 1930 to accurately model the U.S. population?

[This problem is based on the computation by Verhulst, who in 1845 used the 1790-1840 U.S. population data to predict accurately the U.S. population through the year 1930 (long after his own death, of course).]

More space, if necessary, for problem 2.1.29.

## Section 2.2 - Equilibrium Solutions and Stability

2.2.1 - Find the critical points of the autonomous equation

$$\frac{dx}{dt} = x - 4.$$

Then analyze the sign of the equation to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for  $x(t)$  in terms of  $t$ . Finally, use either the exact solution or a computer-generated slope field to sketch typical solution curves for the given differential equation, and verify visually the stability of each critical point.

More space, if necessary, for problem 2.2.1.

**2.2.10** Find the critical points of the autonomous equation

$$\frac{dx}{dt} = 7x - x^2 - 10.$$

Then analyze the sign of the equation to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for  $x(t)$  in terms of  $t$ . Finally, use either the exact solution or a computer-generated slope field to sketch typical solution curves for the given differential equation, and verify visually the stability of each critical point.

More space, if necessary, for problem 2.2.10.

**2.2.21** Consider the differential equation  $dx/dt = kx - x^3$ .

- (a) If  $k \leq 0$ , show that the only critical value  $c = 0$  of  $x$  is stable.
- (b) If  $k > 0$ , show that the critical point  $c = 0$  is now unstable, but that the critical points  $c = \pm\sqrt{k}$  are stable. Thus the qualitative nature of the solutions changes at  $k = 0$  as the parameter  $k$  increases, and so  $k = 0$  is a bifurcation point for the differential equation with parameter  $k$ .

The plot of all points of the form  $(k, c)$  where  $c$  is a critical point of the equation  $x' = kx - x^3$  is the “pitchfork diagram” shown in figure 2.2.13 of the textbook.



More space, if necessary, for problem 2.2.21.

**2.2.23** Suppose that the logistic equation  $dx/dt = kx(M-x)$  models a population  $x(t)$  of fish in a lake after  $t$  months during which no fishing occurs. Now suppose that, because of fishing, fish are removed from the lake at a rate of  $hx$  fish per month (with  $h$  a positive constant). Thus fish are “harvested” at a rate proportional to the existing fish population, rather than at the constant rate of Example 4 from the textbook.

- (a) If  $0 < h < kM$ , show that the population is still logistic. What is the new limiting population?
- (b) If  $h \geq kM$ , show that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ , so the lake is eventually fished out.

More space, if necessary, for problem 2.2.23.

**2.2.24** Separate variables in the logistic harvesting equation

$$dx/dt = k(N - x)(x - H)$$

and then use partial fractions to derive the solution given in equation 15 of the textbook (also appearing in the lecture notes).

More space, if necessary, for problem 2.2.24.

## Section 2.3 - Acceleration-Velocity Models

2.3.1 The acceleration of a Maserati is proportional to the difference between 250 km/h and the velocity of this sports car. If the machine can accelerate from rest to 100 km/h in 10s, how long will it take for the car to accelerate from rest to 200 km/h?

More space, if necessary, for problem 2.3.1.

**2.3.2** Suppose that a body moves through a resisting medium with resistance proportional to its velocity  $v$ , so that  $dv/dt = -kv$ .

(a) Show that its velocity and position at time  $t$  are given by

$$v(t) = v_0 e^{-kt}$$

and

$$x(t) = x_0 + \left(\frac{v_0}{k}\right) (1 - e^{-kt}).$$

(b) Conclude that the body travels only a finite distance, and find that distance.



More space, if necessary, for problem 2.3.2.

**2.3.4** Consider a body that moves horizontally through a medium whose resistance is proportional to the *square* of the velocity  $v$ , so that

$$dv/dt = -kv^2.$$

Show that

$$v(t) = \frac{v_0}{1 + v_0kt}$$

and that

$$x(t) = x_0 + \frac{1}{k} \ln(1 + v_0kt).$$

Note that, in contrast with the result of Problem 2,  $x(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Which offers less resistance when the body is moving fairly slowly - the medium in this problem or the one in Problem 2? Does your answer seem to be consistent with the observed behaviors of  $x(t)$  as  $t \rightarrow \infty$ ?

More space, if necessary, for problem 2.3.4.

**2.3.10** A woman bails out of an airplane at an altitude of 10,000 ft, falls freely for 20s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance  $\rho v$  ft/s<sup>2</sup>, taking  $\rho = .15$  without the parachute and  $\rho = 1.5$  with the parachute. (*Suggestion:* First determine her height above the ground and velocity when the parachute opens.)

More space, if necessary, for problem 2.3.10.

**2.3.24** The mass of the sun is 329,320 times that of the earth and its radius is 109 times the radius of the earth.

- (a) To what radius (in meters) would the earth have to be compressed in order for it to become a *black hole* - the escape velocity from its surface equal to the velocity  $c = 3 \times 10^8 m/s$  of light?
- (b) Repeat part (a) with the sun in place of the earth.