Math 2280 - Assignment 2

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Section 1.4 - 1, 3, 17, 19, 31, 35, 53, 68

Section 1.5 - 1, 15, 21, 29, 38, 42

Section 1.6 - 1, 3, 13, 16, 22, 26, 31, 36, 56

Section 1.4 - Separable Equations and Applications

1.4.1 Find the general solution (implicit if necessary, explicit if convenient) to the differential equation

$$\frac{dy}{dx} + 2xy = 0$$

1.4.3 Find the general solution (implicit if necessary, explicit if convenient) to the differential equation

$$\frac{dy}{dx} = y \sin x$$

1.4.17 Find the general solution (implicit if necessary, explicit if convenient) to the differential equation

$$y' = 1 + x + y + xy.$$

Primes denote the derivatives with respect to x. (Suggestion: Factor the right-hand side.)

1.4.19 Find the explicit particular solution to the initial value problem

$$\frac{dy}{dx} = ye^x, \qquad y(0) = 2e.$$

1.4.31 Discuss the difference between the differential equations $(dy/dx)^2 = 4y$ and $dy/dx = 2\sqrt{y}$. Do they have the same solution curves? Why or why not? Determine the points (a, b) in the plane for which the initial value problem $y' = 2\sqrt{y}$, $y(a) = b$ has **(a)** no solution, **(b)** a unique solution, **(c)** infinitely many solutions.

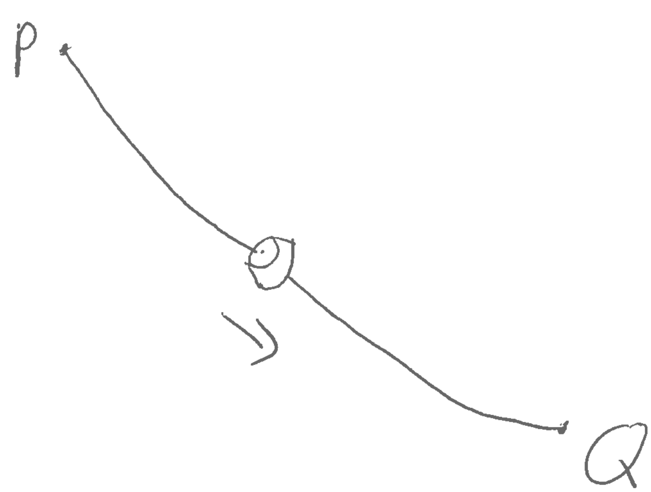
1.4.35 (Radiocarbon dating) Carbon extracted from an ancient skull contained only one-sixth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

1.4.53 Thousands of years ago ancestors of the Native Americans crossed the Bering Strait from Asia and entered the western hemisphere. Since then, they have fanned out across North and South America. The single language that the original Native Americans spoke has since split into many Indian "language families." Assume that the number of these language families has been multiplied by 1.5 every 6000 years. There are now 150 Native American language families in the western hemisphere. About when did the ancestors of today's Native Americans arrive?

1.4.68 The figure below shows a bead sliding down a frictionless wire from point P to point Q. The *brachistochrone problem* asks what shape the wire should be in order to minimize the bead's time of descent from P to Q. In June of 1696, John Bernoulli proposed this problem as a public challenge, with a 6-month deadline (later extended to Easter 1697 at George Leibniz's request). Isaac Newton, then retired from academic life and serving as Warden of the Mint in London, received Bernoulli's challenge on January 29, 1697. The very next day he communicated his own solution - the curve of minimal descent time is an arc of an inverted cycloid - to the Royal Society of London. For a modern derivation of this result, suppose the bead starts from rest at the origin P and let $y = y(x)$ be the equation of the desired curve in a coordinate system with the y-axis pointing downward. Then a mechanical analogue of Snell's law in optics implies that

$$\frac{\sin\alpha}{v} = constant$$

where α denotes the angle of deflection (from the vertical) of the tangent line to the curve - so $\cos\alpha = y'(x)$ (why?) - and $v = \sqrt{2gy}$ is the bead's velocity when it has descended a distance y vertically (from $KE = \frac{1}{2}mv^2 = mgy = -PE$).



(a) First derive from $\sin\alpha/v = constant$ the differential equation

$$\frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}$$

where a is an appropriate positive constant.

(b) Substitute $y = 2a\sin^2 t, dy = 4a \sin t \cos t dt$ in the above differential equation to derive the solution

$$x = a(2t - \sin 2t), \qquad y = a(1 - \cos 2t)$$

for which $t = y = 0$ when $x = 0$. Finally, the substitution of $\theta = 2a$ in the equations for x and y yields the standard parametric equations $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ of the cycloid that is generated by a point on the rim of a circular wheel of radius a as it rolls along the x-axis.

Section 1.5 - Linear First-Order Equations

1.5.1 Find the solution to the initial value problem

$$y' + y = 2 \qquad\qquad y(0) = 0$$

1.5.15 Find the solution to the initial value problem

$$y' + 2xy = x, \qquad y(0) = -2.$$

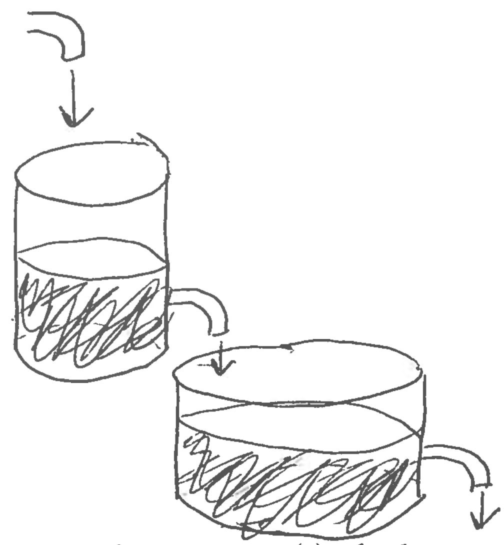
1.5.21 Find the solution to the initial value problem

$$xy' = 3y + x^4 \cos x, \qquad y(2\pi) = 0.$$

1.5.29 Express the general solution of $dy/dx = 1 + 2xy$ in terms of the *error function*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

1.5.38 Consider the *cascade* of two tanks shown below with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lbs of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1.



(a) Find the amount $x(t)$ of salt in tank 1 at time t.

(b) Suppose that $y(t)$ is the amount of salt in tank 2 at time t. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}.$$

and then solve for $y(t)$, using the function $x(t)$ found in part (a).

(c) Finally, find the maximum amount of salt ever in tank 2.

1.5.42 Suppose that a falling hailstone with density $\delta = 1$ starts from rest with negligible radius $r = 0$. Thereafter its radius is $r = kt$ (k is a constant) as it grows by accreation during its fall. Use Newton's secon d law - according to which the net force F acting on a possibly variable mass m equals the time rate of change dp/dt of its momentum $p = mv$ - to set up and solve the initial value problem

$$\frac{d}{dt}(mv) = mg, \qquad v(0) = 0,$$

where m is the variable mass of the hailstone, $v = dy/dt$ is its velocity, and the positive y-axis points downward. Then show that $dv/dt = g/4$. Thus the hailstone falls as though it were under *one-fourth* the influence of gravity.

Section 1.6 - Substitution Methods and Exact Equations

1.6.1 Find the general solution of the differential equation

$$(x + y)y' = x - y$$

1.6.3 Find the general solution of the differential equation

$$xy' = y + 2\sqrt{xy}$$

1.6.13 Find the general solution of the differential equation

$$xy' = y + \sqrt{x^2 + y^2}$$

1.6.16 Find the general solution of the differential equation

$$y' = \sqrt{x + y + 1}$$

1.6.22 Find the general solution of the differential equation

$$x^2y' + 2xy = 5y^4$$

1.6.26 Find the general solution of the differential equation

$$2y^2y' + y^3 = e^{-x}$$

1.6.31 Verify that the differential equation

$$(2x + 3y)dx + (3x + 2y)dy = 0$$

is exact; then solve it.

1.6.36 Verify that the differential equation

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$$

is exact; then solve it.

1.6.56 Suppose that $n \neq 0$ and $n \neq 1$. Show that the substitutuion $v = y^{1-n}$ transforms the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

into the linear equation

$$\frac{dv}{dx} + (1-n)P(x)v(x) = (1-n)Q(x).$$