Math 2280 - Project 2 Writeup

Dylan Zwick

Spring 2009

This is my sample writeup for our second class project. Your writeup obviously doesn't need to be exactly like this, but it should contain more or less the same information.

1 Derivation of Kepler's First and Second Laws from Newton's Law of Universal Gravitation

Kepler's three laws of planetary motion state:

1. The orbit of each planet is an ellipse with the sun at one focus.
2. The radius vector from the sun to each planet sweeps out area at a constant rate.
3. The square of the planet's period of revolution is proportional to the cube of the major semiaxis of its elliptical orbit.

In Newton's great book *Principia Mathematica* (1687) he derived his law of universal gravitation based upon Kepler's laws. In this project we're going to go the other way, essentially proving that Newton's law of universal gravitation is equivalent to Kepler's laws, in terms of planetary motion at least.

1.1 Preliminaries

For this derivation we will assume that the sun is located at the origin in the plane of motion of a planet. We're assuming that we're dealing with a radial force, and so it's OK to assume that the planet's motion takes place in a plane. In this case we can write the position vector of the planet in cartesian form as:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

where **i** and **j** are the unit vectors in the x and y coordinates, respectively.

Newton's law of universal gravitation states that:

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\mathbf{u}.$$

Where **F** is the force upon the orbiting planet, G is Newton's universal gravitational constant, m_1 is the mass of the sun, m_2 is the mass of the planet, $r = \sqrt{x^2+y^2}$ is the distance from the sun to the planet, and **u** is a unit vector that points from the sun at the origin towards the planet. We note that G and m_1 are both constants, and so we can write their product as k. We also define $\mathbf{r} = r\mathbf{u}$. We finally note that Newton's second law of motion states that $\mathbf{F} = m_2\mathbf{r}''$. If we make these substitutions and simplify we get the relation:

$$\mathbf{r}'' = -\frac{k\mathbf{r}}{r^3}.$$

If we switch to polar coordinates, then the radial and transverse unit vectors are given by:

$$\mathbf{u}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

and

$$\mathbf{u}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}.$$

Here the radial unit vector $\mathbf{u}_r$ always points directly away from the origin, while the transverse unit vector $\mathbf{u}_\theta$ is rotated 90° from the radial unit vector.

1.2 Polar Relations

If we differentiate our equations for the radial and transverse unit vectors we get:

$$\frac{d\mathbf{u}_r}{dt} = -\sin\theta\frac{d\theta}{dt}\mathbf{i} + \cos\theta\frac{d\theta}{dt}\mathbf{j} = \mathbf{u}_\theta\frac{d\theta}{dt}.$$

and

$$\frac{d\mathbf{u}_\theta}{dt} = -\cos\theta\frac{d\theta}{dt}\mathbf{i} - \sin\theta\frac{d\theta}{dt}\mathbf{j} = -\mathbf{u}_r\frac{d\theta}{dt}.$$

Now, if we differentate our radial vector $\mathbf{r} = r\mathbf{u}_r$ we get, according to the product rule:

$$\frac{d\mathbf{r}}{dt} = r\frac{d\mathbf{u}_r}{dt} + \frac{dr}{dt}\mathbf{u}_r = r\frac{d\theta}{dt}\mathbf{u}_\theta + \frac{dr}{dt}\mathbf{u}_r.$$

Now, if we differentiate this one more time we get:

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= r\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{dt} + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt} + \frac{d^2r}{dt^2}\mathbf{u}_r \\ &= -r\left(\frac{d\theta}{dt}\right)^2\mathbf{u}_r + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + \frac{d^2r}{dt^2}\mathbf{u}_r \\ &= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta.\end{aligned}$$

If we then note that

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}$$

then we see that the above finally simplifies to:

$$\mathbf{r}'' = \left[\frac{d^2}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\mathbf{u}_\theta.$$

Now, if we use Newton's law of universal gravitation we see that we must have the equality:

$$\mathbf{r}'' = -\frac{k}{r^2}\mathbf{u}_r = \left[\frac{d^2}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\mathbf{u}_\theta.$$

If we equate the direction vectors we get the two relations:

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

and

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2}.$$

It is from these two equations that we can derive Kepler's first and second laws of planetary motion.

1.3 Kepler's Second Law

The first of these equations implies that:

$$r^2\frac{d\theta}{dt} = h$$

where h is a constant. The polar-coordinate area element is $dA = \frac{1}{2}r^2 d\theta$, and so the above relation implies that $dA/dt = 2h$, and as $2h$ is a constant we have that this is Kepler's second law.

1.4 Kepler's First Law

To derive Kepler's first law of, we want to equate the radial components of the two expressions we derived for $\mathbf{r}'' = \mathbf{a}$:

$$\mathbf{r}'' = -\frac{k}{r^3}\mathbf{r} = -\frac{k}{r^2}\mathbf{u}_r = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r.$$

If we combine this with our earlier relation:

$$r^2\frac{d\theta}{dt} = h$$

which implies

$$r\left(\frac{d\theta}{dt}\right)^2 = \frac{h^2}{r^3}$$

we get the equivalence:

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{k}{r^2}.$$

Now, this is a differential equation, but it's nonlinear. However, if we make the substitution $r = 1/z$ we get:

$$\frac{dr}{dt} = -\frac{1}{z^2}\frac{dz}{dt} = -r^2\frac{dz}{d\theta}\frac{d\theta}{dt} = -h\frac{dz}{d\theta}.$$

If we differentiate this again with respect to t we get:

$$\frac{d^2r}{dt^2} = -h\frac{d^2z}{d\theta^2}\frac{d\theta}{dt}.$$

If we then use our earlier equality:

$$\frac{d\theta}{dt} = \frac{h}{r^2}$$

we get:

$$\frac{d^2r}{dt^2} = -\frac{h^2}{r^2}\frac{d^2z}{d\theta^2}.$$

If we then plug this relation into our equality:

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{k}{r^2}$$

we get:

$$-\frac{h^2}{r^2}\frac{d^2z}{d\theta^2} - \frac{h^2}{r^3} = -\frac{k}{r^2}.$$

If we then multiply through by $-r^2/h^2$ we get:

$$\frac{d^2z}{d\theta^2} + \frac{1}{r} = \frac{k}{h^2}.$$

If we then finally use our defining relation for z, namely $z = 1/r$, we get:

$$\frac{d^2z}{d\theta^2} + z = \frac{k}{h^2},$$

which is a linear differential equation we know how to solve!

The corresponding homogeneous equation is:

$$\frac{d^2z}{d\theta^2} + z = 0$$

which has solution:

$$z_h(\theta) = A\sin\theta + B\cos\theta.$$

Using the method of undetermined coefficients we get that our particular solution is:

$$z_p(\theta) = \frac{k}{h^2}.$$

Therefore, our general solution is:

$$z(\theta) = z_h(\theta) + z_p(\theta) = A\sin\theta + B\cos\theta + \frac{k}{h^2}.$$

Now, if we define the constants α, e, and L to be:

$$\alpha = \arctan\left(\frac{A}{B}\right)$$
$$e = \frac{h^2\sqrt{A^2+B^2}}{k}$$
$$L = \frac{h^2}{k}$$

then we note that:

$$A = \sqrt{A^2+B^2}\sin\alpha$$
$$B = \sqrt{A^2+B^2}\cos\alpha$$

and so our formula becomes:

$$(\sqrt{A^2+B^2})(\sin\alpha\sin\theta + \cos\alpha\cos\theta) + \frac{1}{L}.$$

Now, if we use the trigonometric relation:

$$\cos(\phi - \psi) = \sin\phi \sin\psi + \cos\phi \cos\psi$$

then we get for our formula above:

$$(\sqrt{A^2 + B^2}) \cos(\theta - \alpha) + \frac{1}{L}.$$

Finally, if we note that:

$$\sqrt{A^2 + B^2} = \sqrt{A^2 + B^2}\left(\frac{h^2}{k^2}\right)\left(\frac{k^2}{h^2}\right) = \left(\frac{h^2\sqrt{A^2 + B^2}}{k^2}\right)\left(\frac{1}{L}\right) = \frac{e}{L},$$

then our above formula becomes:

$$\frac{e\cos(\theta - \alpha)}{L} + \frac{1}{L} = \frac{1 + e\cos(\theta - \alpha)}{L}.$$

So, we have:

$$z(\theta) = \frac{1 + e\cos(\theta - \alpha)}{L}$$

and therefore given $r = 1/z$:

$$r(\theta) = \frac{L}{1 + e\cos(\theta - \alpha)}.$$

This is the polar formula for an ellipse. Therefore, the motion of the planet around the sun is an ellipse, and this is Kepler's first law.

1.5 Plotting

As an example of some elliptical orbits, attached are some ellipses plotted in Maple. They were plotted using a parametric plotter, using the parametric relations:

$$x(t) = r(t)\cos t,\ y(t) = r(t)\sin t,\ 0 \le t \le 2\pi.$$

The parametric formulas for an ellipse are:

$x := (t) \rightarrow \dfrac{L \cdot \cos(t)}{1 + e \cdot \cos(t - a)};$

$$t \rightarrow \frac{L \cos(t)}{1 + e \cos(t - a)} \tag{1}$$

$y := (t) \rightarrow \dfrac{L \cdot \sin(t)}{1 + e \cdot \cos(t - a)};$

$$t \rightarrow \frac{L \sin(t)}{1 + e \cos(t - a)} \tag{2}$$

For a simple graph of a unit circle we enter:

$L := 1;$

$$1 \tag{3}$$

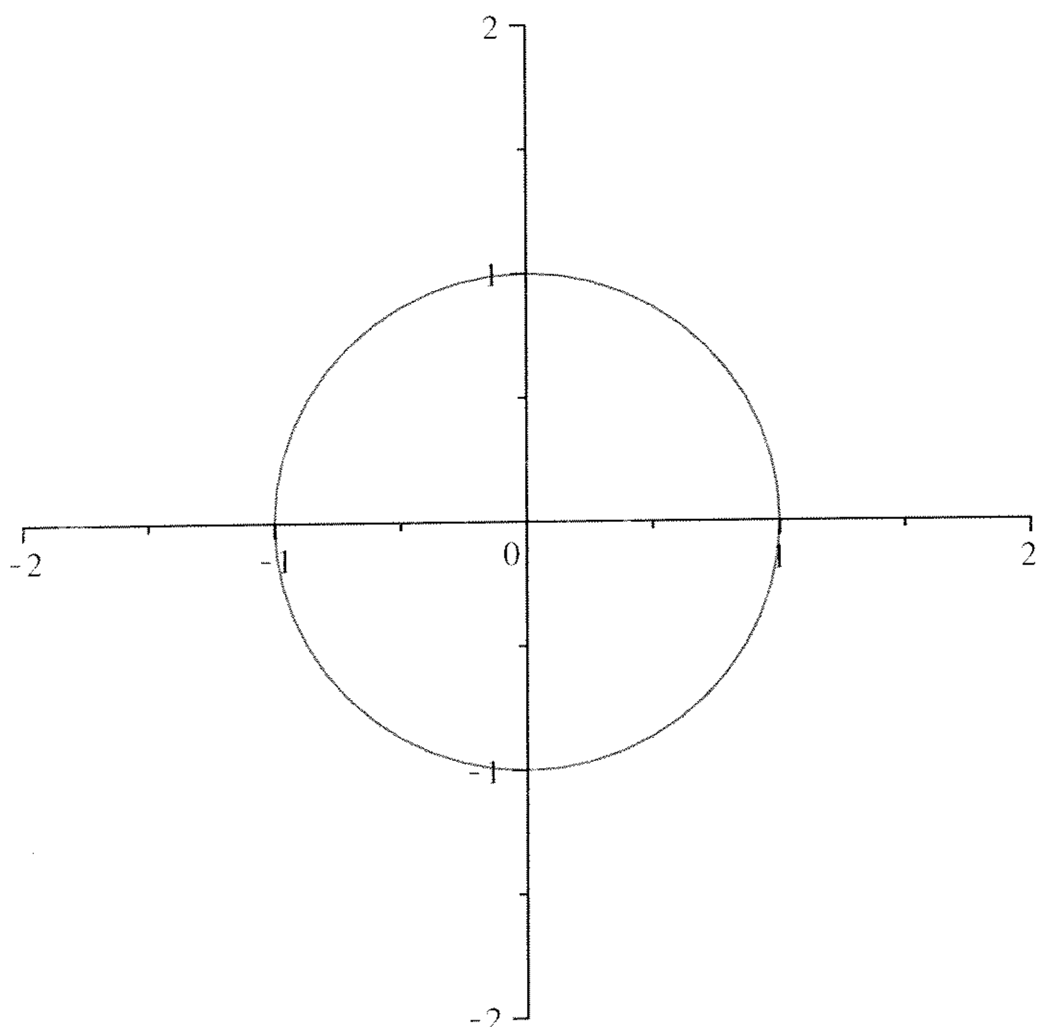
$e := 0;$

$$0 \tag{4}$$

$a := 0;$

$$0 \tag{5}$$

$plot([x(t), y(t), t = 0..2 \cdot \mathrm{Pi}], -2..2, -2..2);$

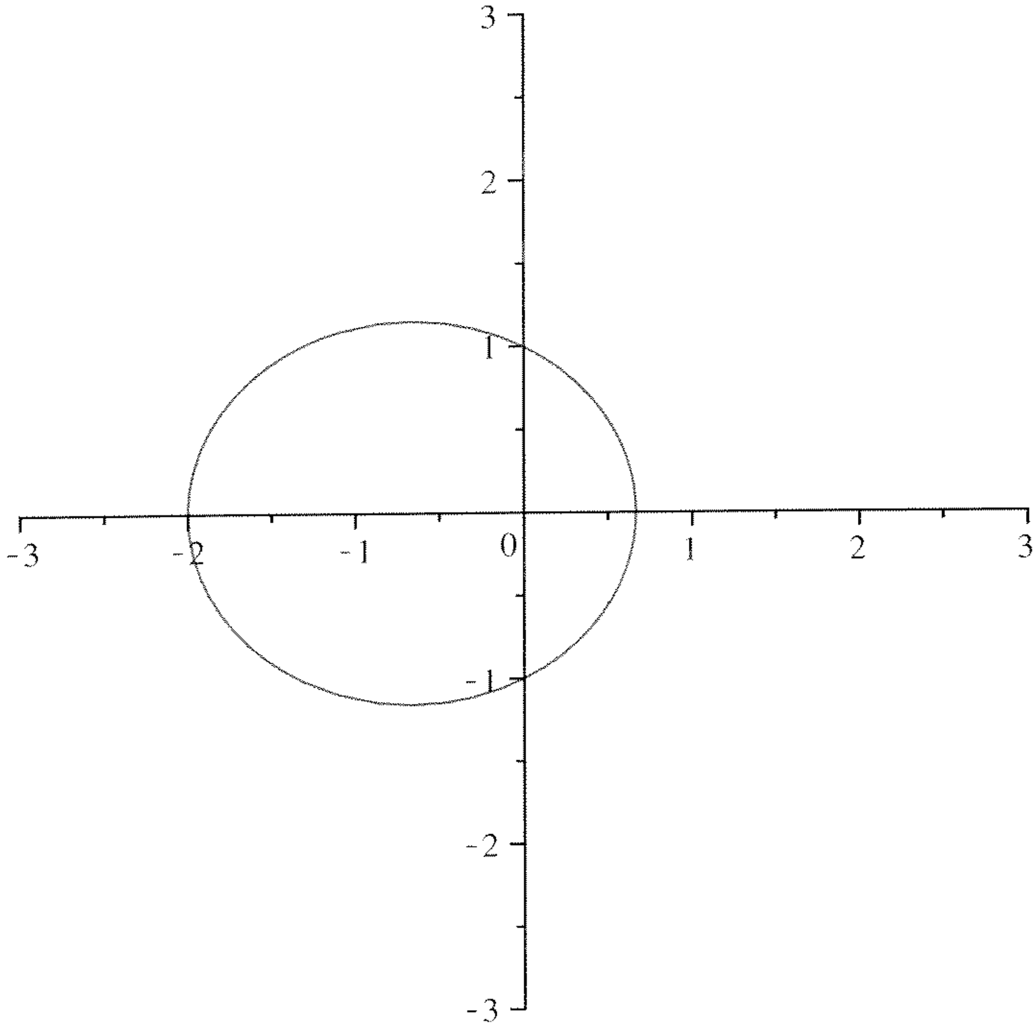


Taking a look at some other ellipses:

$e := .5;$

$$0.5 \tag{6}$$

$plot([x(t), y(t), t = 0..2 \cdot \text{Pi}], -3..3, -3..3);$

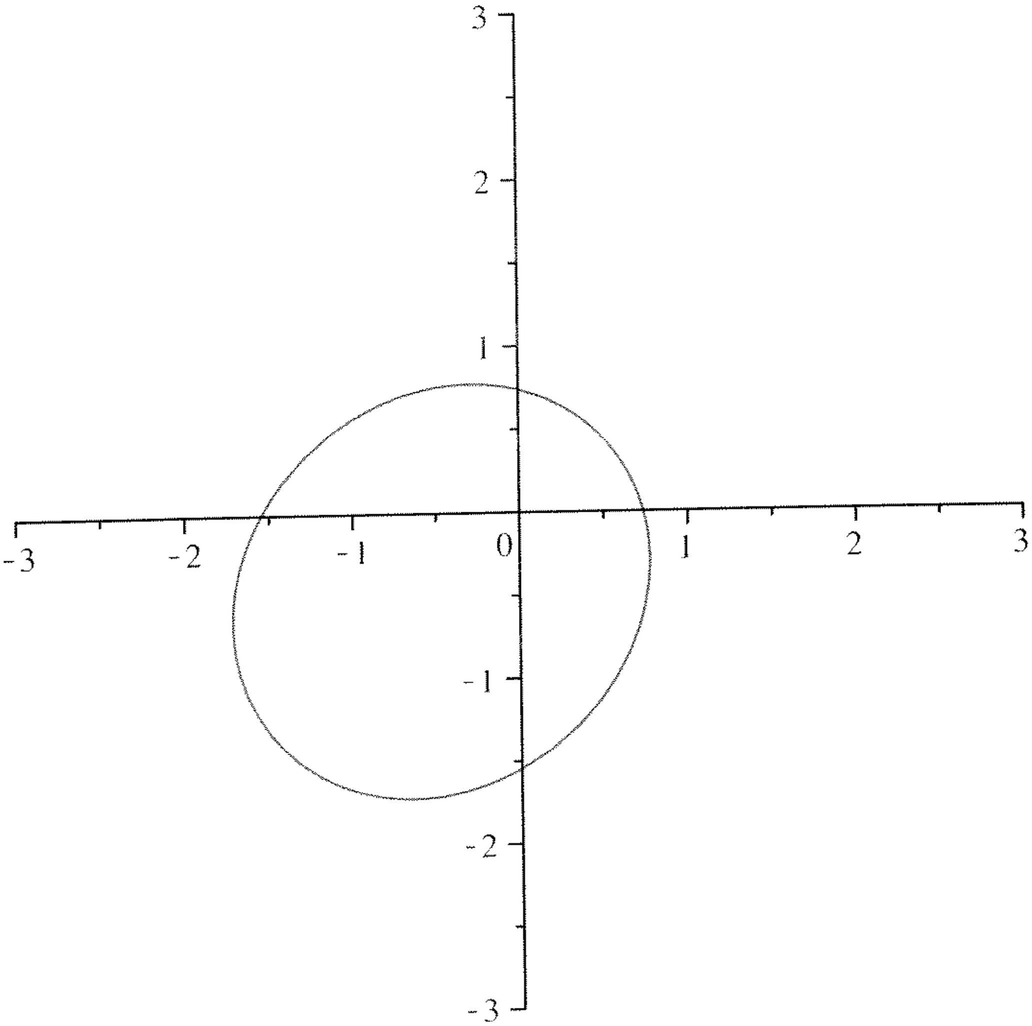


Rotating this ellipse:

$a := \frac{\text{Pi}}{4};$

$$\frac{1}{4}\pi \tag{7}$$

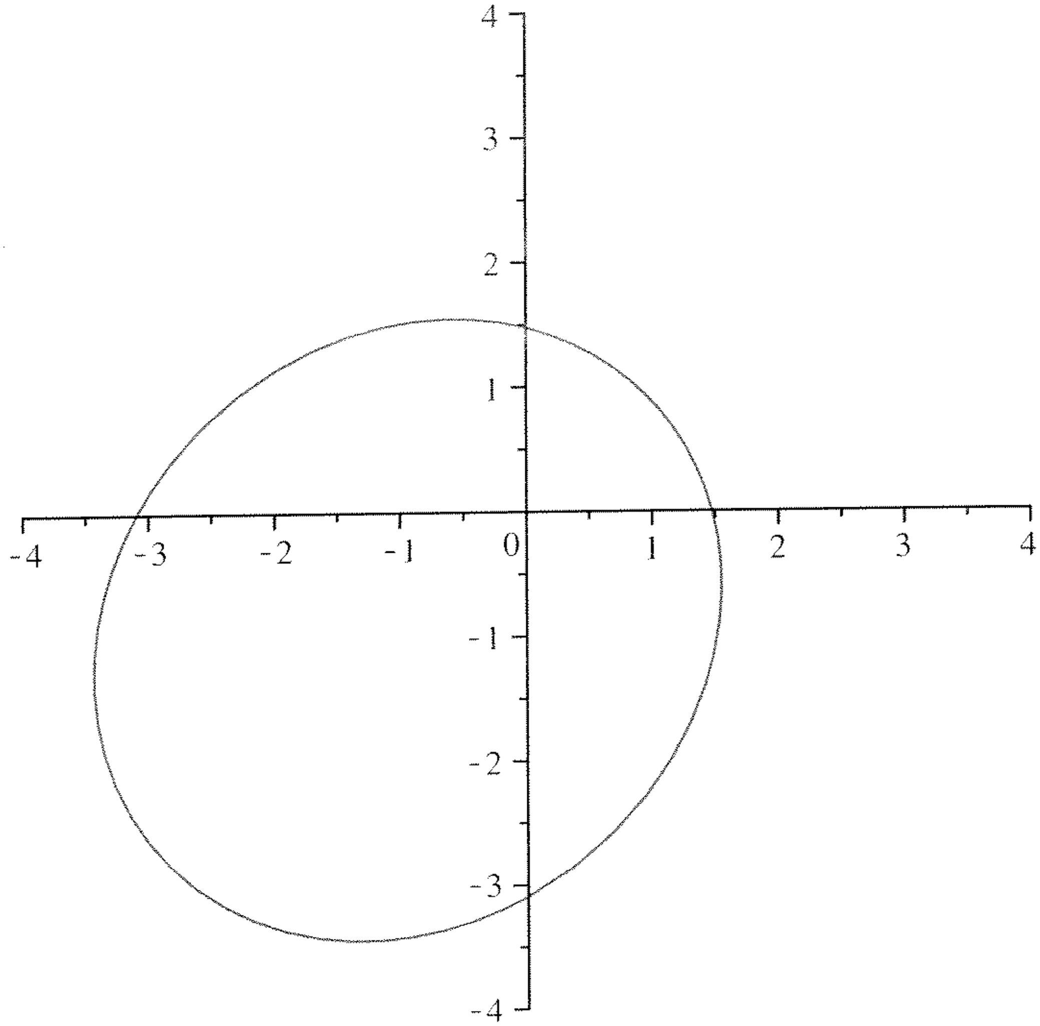
$plot([x(t), y(t), t = 0..2 \cdot \pi], -3..3, -3..3);$



Increasing its size:
$L := 2;$

$$2 \tag{8}$$

$plot([x(t), y(t), t = 0..2\cdot\pi], -4..4, -4..4);$



Let's take a look at the shape of the orbits for some of the planets in the solar system. Here's we'll normalize things so L = 1 and a = 0.

$L := 1;$

$$1 \tag{9}$$

$a := 0;$

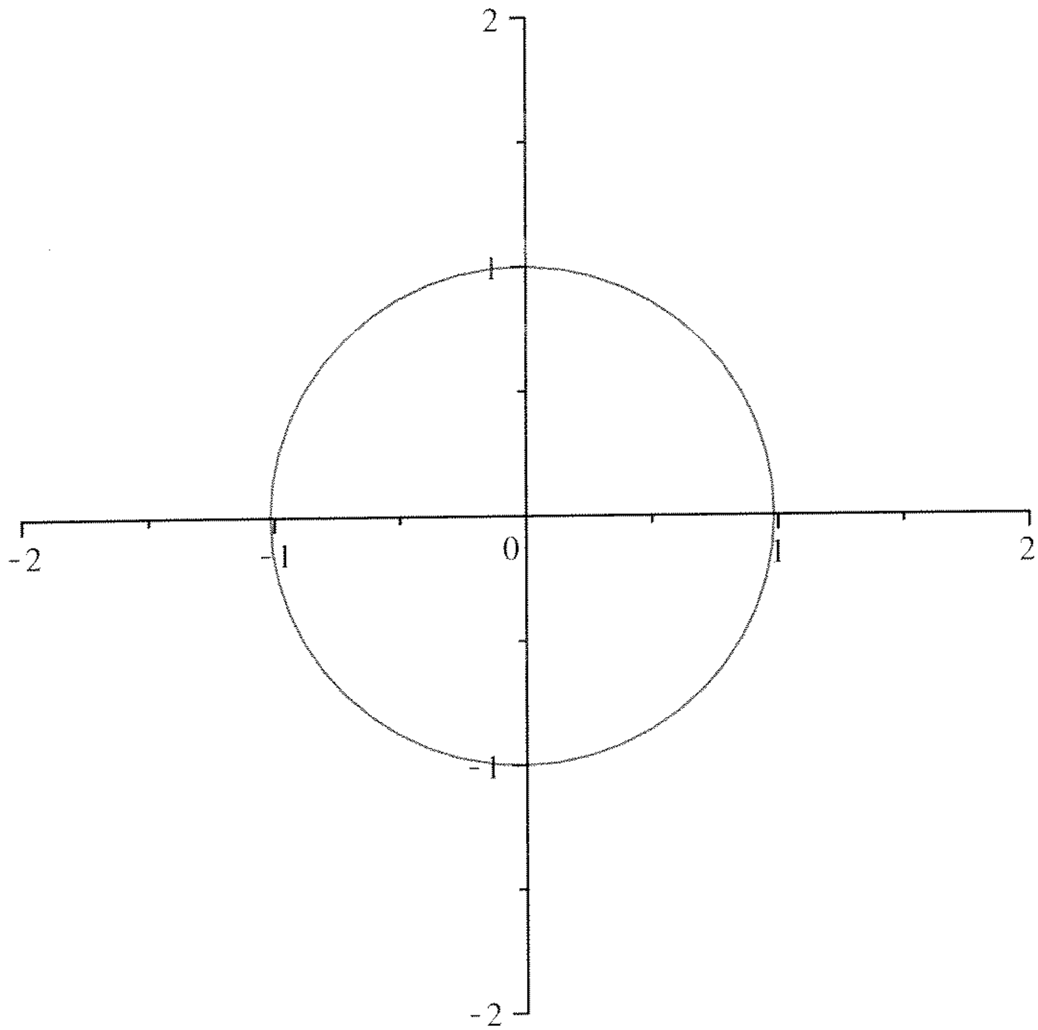
$$0 \tag{10}$$

For the Earth we have:

$e := .0167;$

$$0.0167 \tag{11}$$

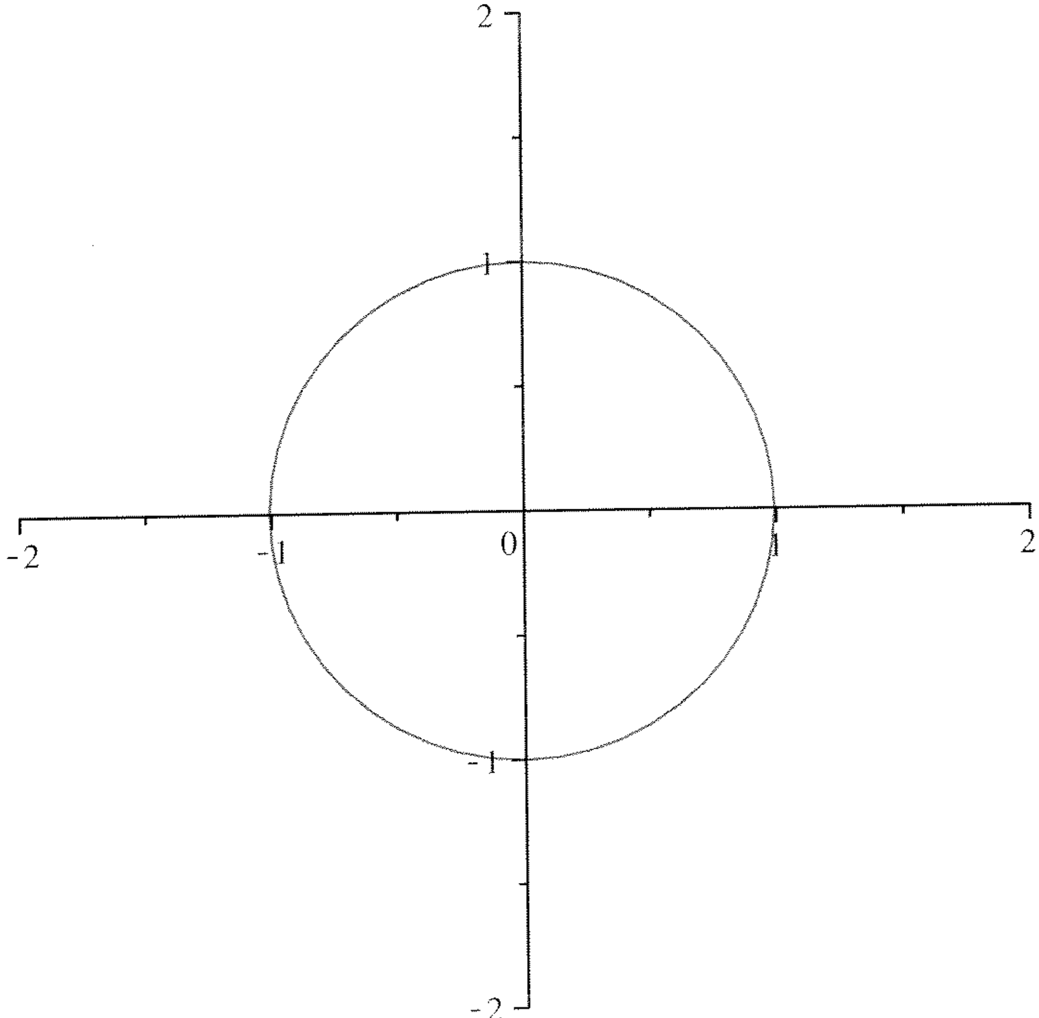
$plot\left(\left[x(t), y(t), t = 0..2\cdot\pi\right], -2..2, -2..2\right);$



Which is very close to a circular orbit! No wonder Copernicus thought the orbits were circular. As for the other planets, for Venus we have an even less eccentric orbit:
$e := .0068;$

$$0.0068 \tag{12}$$

$plot\left(\left[x(t), y(t), t=0..2\cdot\pi\right], -2..2, -2..2\right);$

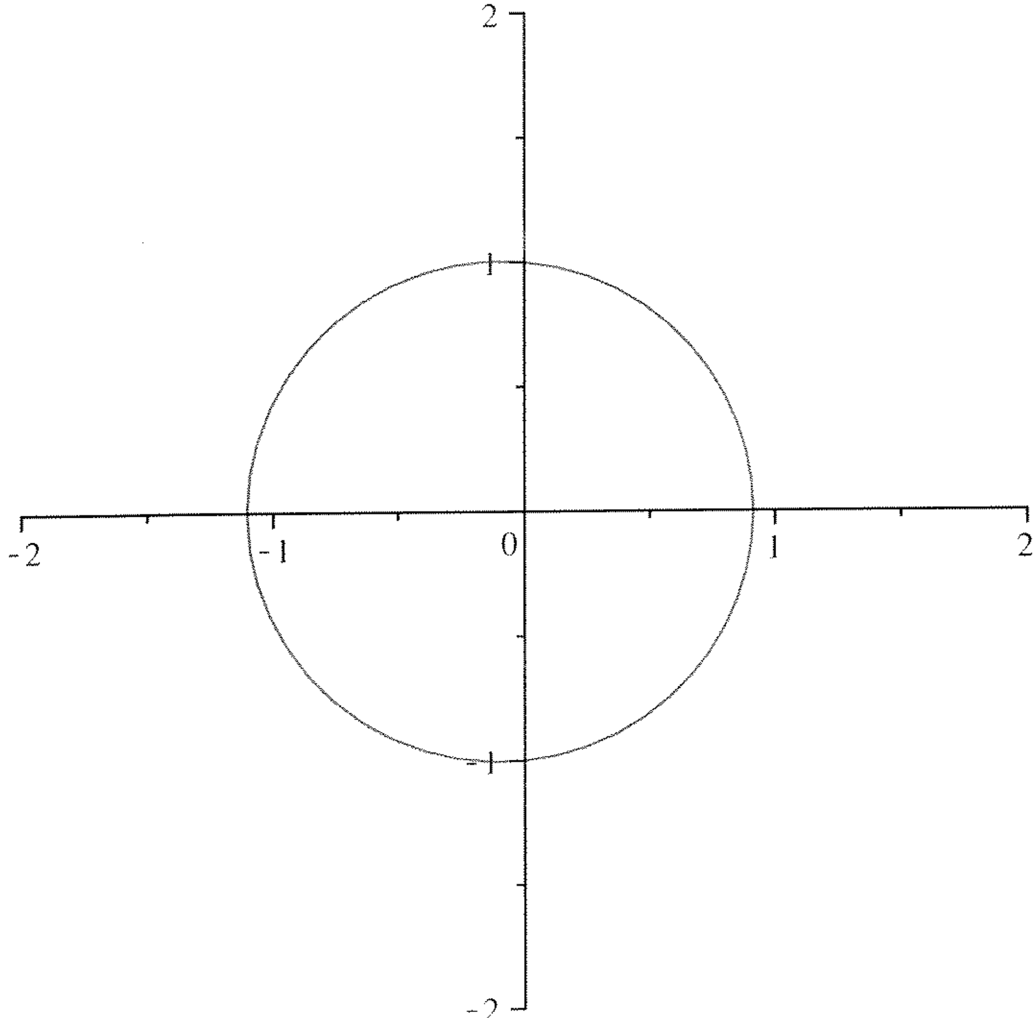


While for Mars we have a slightly more eccentric orbit, but still pretty close to circular:

$e := .0933;$

$$0.0933 \tag{13}$$

$plot([x(t), y(t), t = 0..2\cdot\pi], -2..2, -2..2);$

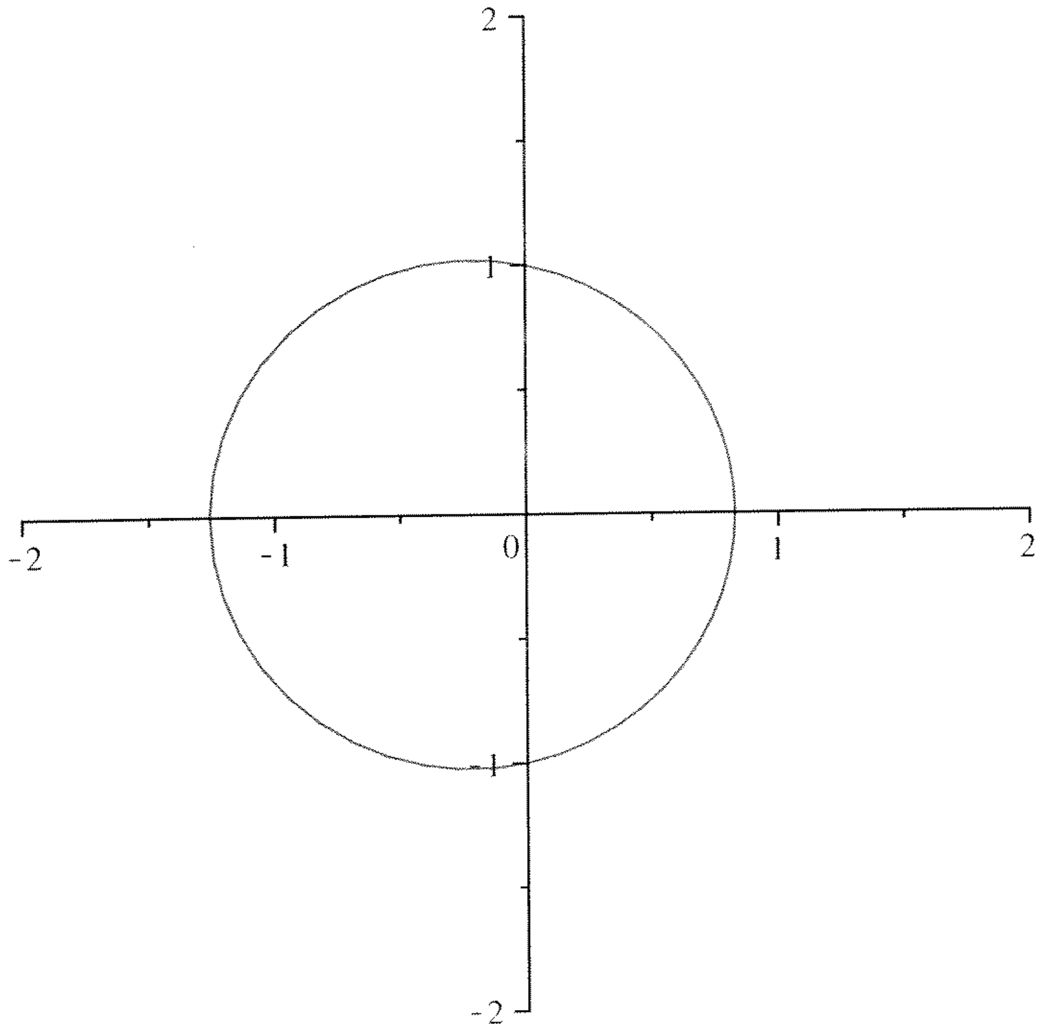


While for Mercury we have a substantially more eccentric orbit:

$e := .2056$

$$0.2056 \tag{14}$$

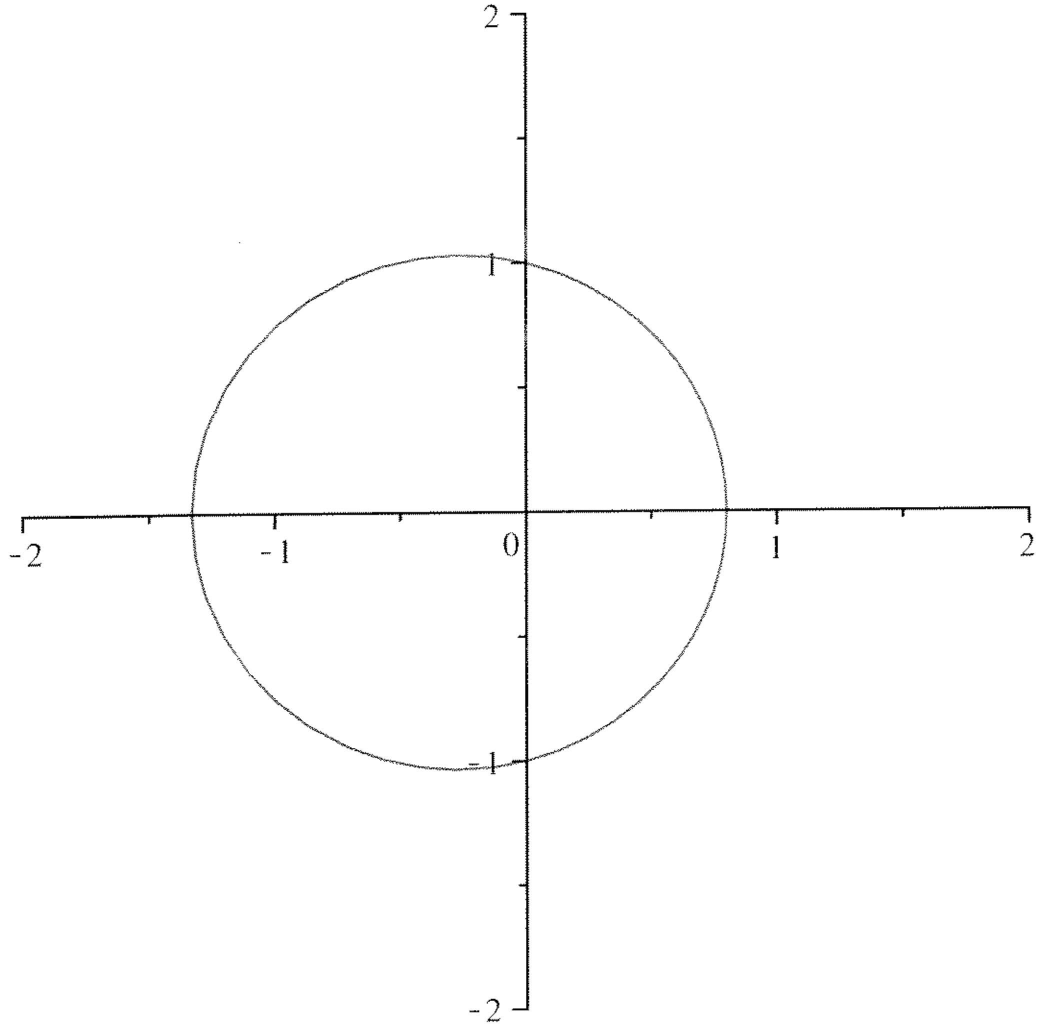
$plot([x(t), y(t), t = 0..2\cdot\pi], -2..2, -2..2);$



And for Pluto the orbit is more eccentric still:
$e := .2486;$

$$0.2486 \tag{15}$$

$plot\left(\left[x(t), y(t), t = 0 .. 2 \cdot \pi\right], -2 .. 2, -2 .. 2\right);$

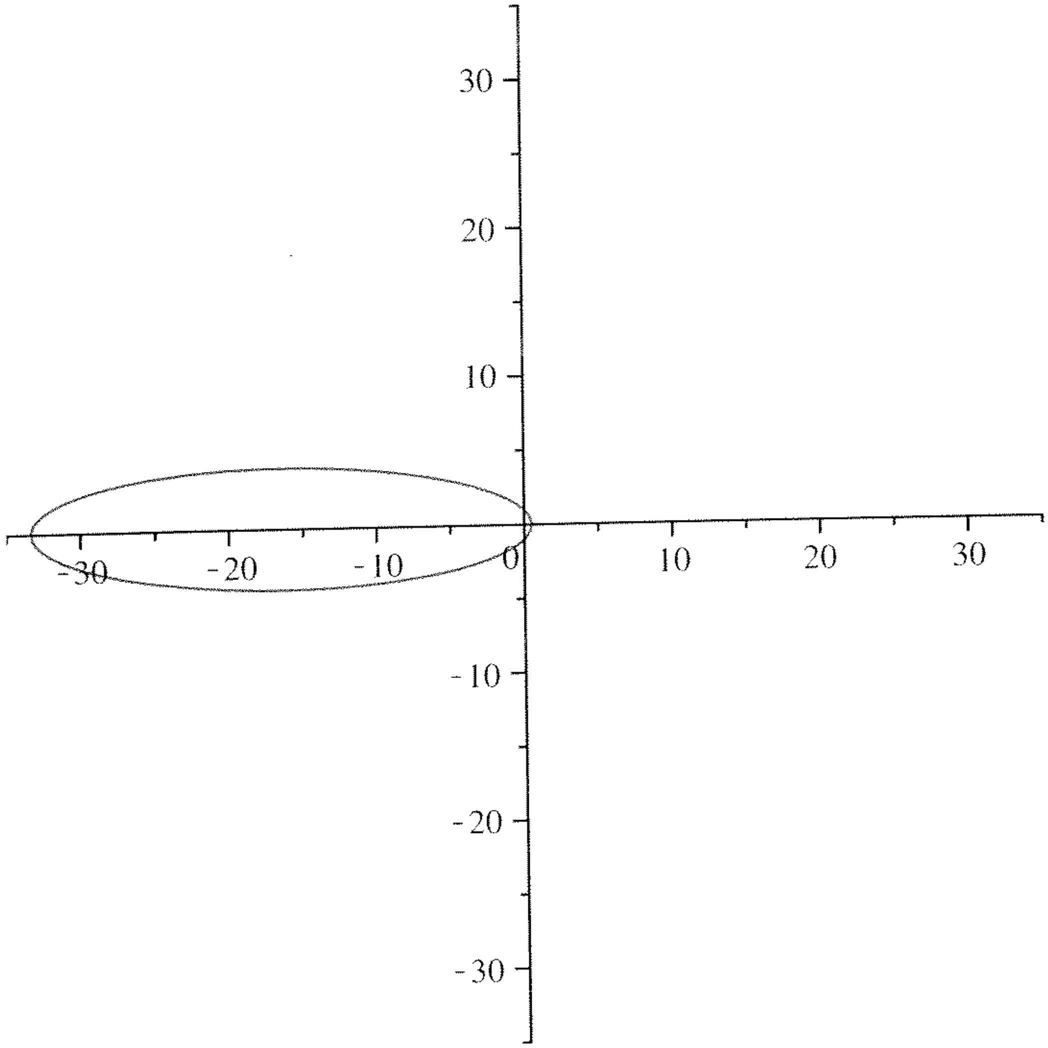


All of these are reasonably circular, and so none of the planets have an exceptionally elliptical orbit. Otherwise, they might smash into each other! On the other hand, comets frequently have VERY eccentric orbits. Haley's comet for example has a very high eccentricity, and so a very noncircular orbit:

$e := .97$

$$0.97 \tag{16}$$

$plot([x(t), y(t), t = 0..2\cdot\pi], -35..35, -35..35);$



Where here, as is the case for all of these ellipses, the sun is at the origin.