Math 2280 - Explanation for Step 6

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Class, there has been some confusion about how we handle step 6 of the project, as it requires some chain rule jiu jitsu. So, here's an explanation of how it works.

If we define $r = 1/z$ then the chain rule tells us that:

$$\frac{dr}{dt} = \frac{dr}{dz}\frac{dz}{dt} = -\frac{1}{z^2}\frac{dz}{dt}.$$

If we then use the relation from the textbook:

$$r^2\frac{d\theta}{dt} = h$$

we get:

$$-\frac{1}{z^2}\frac{dz}{dt} = -r^2\frac{dz}{dt} = -h\frac{dt}{d\theta}\frac{dz}{dt} = -h\frac{dz}{d\theta}.$$

And so we derive the relation:

$$\frac{dr}{dt} = -h\frac{dz}{d\theta}.$$

If we differentiate again with respect to t and again use the chain rule we get:

$$\frac{d^2r}{dt^2} = -h\frac{d^2z}{d\theta^2}\frac{d\theta}{dt}.$$

Now, if we agin use our relation:

$$r^2\frac{d\theta}{dt} = h$$

then we get:

$$\frac{d^2r}{dt^2} = -h\frac{d^2z}{d\theta^2}\frac{h}{r^2} = -\frac{h^2}{r^2}\frac{d^2z}{d\theta^2}.$$

If we then equate this with our relation from the textbook:

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{k}{r^2}$$

we get:

$$-\frac{h^2}{r^2}\frac{d^2z}{d\theta^2} - \frac{h^2}{r^3} = -\frac{k}{r^2}$$

which simplifies to

$$\frac{d^2z}{d\theta^2} + \frac{1}{r} = \frac{k}{h^2}.$$

If we then use our defining relation for z, namely $z = 1/r$, then we get the relation:

$$\frac{d^2z}{d\theta^2} + z = \frac{k}{h^2}.$$

Which is what we want to derive.