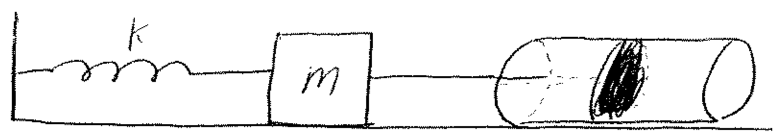
Math 2280 - Lecture 9

Dylan Zwick

Spring 2009

1 Mechanical Vibrations

Today we're going to examine a fairly simple mechanical system in detail, and look closely at its possible solutions.



We have a mass on a spring connected to a dashpot. The forces on the mass are:

$$F_S = -kx$$
$$F_R = -cv$$
$$F_E = \text{external force (today 0)}$$

So, according to Newton's second law:

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx$$

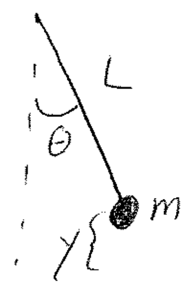
or, after some algebra,

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

This is a second order linear homogeneous ODE with constant coefficients. We can rewrite this as:

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Another basic mechanical example is the simple pendulum:



which, according to the conservation of energy, gives us the differential equation:

$$mgy + \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 = C$$

where, if we note that $y = L(1 - \cos\theta)$ we get:

$$mgL(1 - \cos\theta) + \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 = C$$

and if we differentiate both sides of this we get the relation:

$$mgL\sin\theta\frac{d\theta}{dt} + mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) = 0$$

and if we divide through by the common factors we get:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

and if we assume that θ is small we can use the relation $\sin\theta \approx \theta$ to get:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

which we note is the same thing we saw before with the mass-spring-dashpot system if we set $c = 0$.

This differential equation has the solutions:

$$\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{L}}t\right)$$

If we choose:

$$A = \sqrt{c_1^2 + c_2^2}$$

and

$$\cos o = \frac{c_1}{A}, \sin o = \frac{c_2}{A}$$

then

$$\theta(t) = A\left(\cos o \cos\left(\sqrt{\frac{g}{L}}t\right) + \sin\theta \sin\left(\sqrt{\frac{g}{L}}t\right)\right).$$

If we use the relation:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

then we get the solution:

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{L}}t - o\right).$$

This is the simplified equation for simple harmonic motion. We call the terms:

$$A = \text{amplitude}$$
$$\phi = \text{phase shift}$$
$$\sqrt{\frac{g}{L}} = \text{angular frequency} = \omega.$$

From these we define the terms:

$$\text{Frequency}: f = \frac{\omega}{2\pi}$$
$$\text{Period}: T = \frac{1}{f} = \frac{2\pi}{\omega}.$$

Now, if we look again at the mass-spring-dashpot situation we examined at the beginning of this lecture we note that we can rewrite the differential equation as:

$$x'' + 2px' + \omega_0^2 x = 0$$
$$\text{with}$$
$$\omega_0 = \sqrt{\frac{k}{m}} > 0, \text{ and } p = \frac{c}{2m} > 0.$$

If we use the quadratic equation to solve the characteristic equation for this ODE we get:

$$\frac{-2p \pm \sqrt{(2p)^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

These solutions give us three cases:

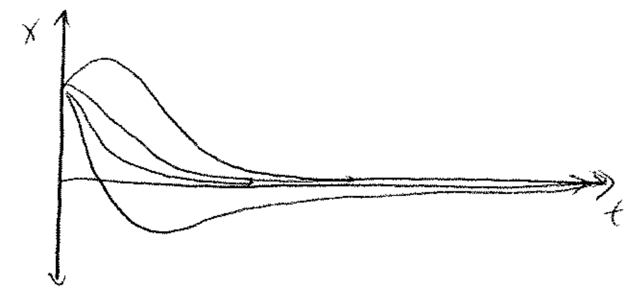
Case 1: Overdamped This case occurs when

$$p > \omega_0 \text{ or } c^2 > 4mk.$$

In this situation we have 2 real negative roots, and our solution is of the form:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Some representative graphs of this situation are below. We note that the solution asymptotically goes to 0 as $t \to \infty$.



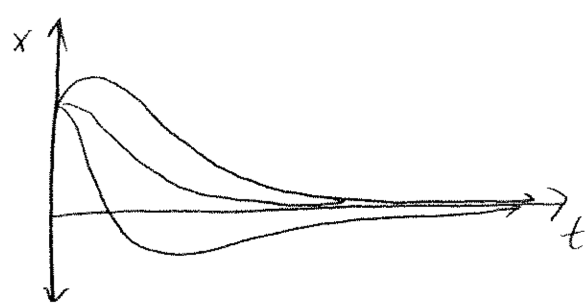
Case 2: Critically Damped This case occurs when

$$p = \omega_0 \text{ or } c^2 = 4mk.$$

In this situation we have one real negative root and our solution is of the form:

$$x(t) = e^{-rt}(c_1 + c_2 t).$$

Some representative graphs of this situation are below. We note that the solution asymptotically goes to 0 as $t \to \infty$.



Case 3: Underdamped This case occurs when

$$p < \omega_0 \text{ or } c^2 < 4km.$$

In this situation we have two complex roots and our solution is of the form:

$$x(t) = e^{-pt}(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t))$$

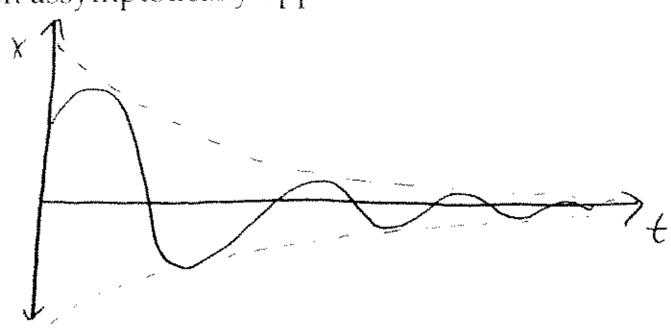
where

$$\omega_1 = \sqrt{\omega_0^2 - p^2} = \frac{\sqrt{4km - c^2}}{2m}.$$

We can rewrite this solution as:

$$x(t) = Ce^{-pt} \cos(\omega_1 t - \alpha).$$

A representative graph of this situation is below. We note, again, that the solution assymptotically approaches 0 as $t \to \infty$ unless $p = 0$.



Example - Solve the ODE that models the mass-spring-dashpot system with the parameters:

$$m = \frac{1}{2}, c = 3, k = 4, x_0 = 2, v_0 = 0.$$

The corresponding ODE would be:

$$\frac{1}{2}x'' + 3x' + 4x = 0$$

with solution:

$$x(t) = c_1 e^{-4t} + c_2 e^{-2t}.$$

If we then use the given initial conditions to figure out the constants c_1 and c_2 we get:

$$x(t) = 4e^{-2t} - 2e^{-4t}.$$

2 Nonhomogeneous Equations

Up to this point we've dealt pretty much exclusively with homogeneous linear equations:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

We're going to take a look at what we get when we add a "driving force" or a nonhomogeneous term to the right side of our equation:

$$a_n y^{(n)} + \cdots + a_0 y = f(x).$$

Now, we learned in general that "all" we had to do was solve the homogeneous equation to get:

$$y_h = c_1 y_1 + \cdots + c_n y_n$$

where the y_k are linearly independent, and then find a particular solution to the nonhomogeneous solution y_p. If you've got these, then the general solution is of the form:

$$y = y_p + y_h.$$

where the c_k in y_h are determined by initial conditions.

Now, finding these solutions is a difficult thing to do, but in a few situations we can simplify things to make our job easier.

Example - If $f(x)$ is a polynomial, guess y_p is also a polynomial of the same degree:

$$f(x) = \text{polynomial of degree } m$$
$$y_p = A_m x^m + \cdots + A_1 x + A_0$$

then, plug in and use linear methods to solve for the A_k.

Example - Find a particular solution to the differential equation:

$$y'' + 3y' + 4y = 3x + 2$$

We guess the solution is of the form: $y = A_1 x + A_0$. If we plug this into the equation we get:

$$3A_1 + 4(A_1 x + A_0) = 3x + 2$$

and then solving for A_1 and A_0 we get:

$$y_p = \frac{3}{4}x - \frac{1}{16}.$$

Now, the same idea holds if $f(x)$ is of the form:

$$f(x) = Ce^{rx}$$

or

$$f(x) = c_1 \sin kx + c_2 \cos kx.$$

In these cases, we guess:

$$y_p = Ae^{rx}$$

$$y_p = A_1 \sin kx + B_1 \cos kx.$$

Now, if $f(x)$ has any of these three types of terms multiplied together, we just guess that our solution has these terms multiplied together. If $f(x)$ has any of these three types of terms added together, we just guess that our solution has these terms added together. Make sure to include all the polynomial terms!

So, for example, if we want to guess the form of the particular solution to the ODE:

$$y^{(3)} + 9y' = x \sin x + x^2 e^{2x}$$

we would guess:

$$y_p = A \cos x + B \sin x + Cx \cos x + Dx \sin x + Ee^{2x} + Fxe^{2x} + Gx^2 e^{2x}$$

and then calculate A, B, C, D, E, F and G appropriately. This is called the method of undetermined coefficients. This method will work in general, except for one issue that we can illustrate with two examples.

Example - Find a particular solution of the ODE:

$$y'' - 4y = 2e^{3x}.$$

We guess $y_p = Ae^{3x}$ and plug this in to the ODE to get:

$$5Ae^{3x} = 2e^{3x}$$

from which we get $A = \frac{2}{5}$. So, our particular solution is:

$$y_p = \frac{2}{5}e^{3x}.$$

The homogeneous solution to this ODE is:

$$y_h = c_1 e^{2x} + c_2 e^{-2x}$$

and so the general form of the solution is:

$$y = \frac{2}{5}e^{3x} + c_1 e^{2x} + c_2 e^{-2x}.$$

No problems there. However, if instead we try to solve the superficially similar ODE:

$$y'' - 4y = 2e^{2x}$$

if we try $y_p = Ae^{2x}$ then we get:

$$4Ae^{2x} - 4Ae^{2x} = 2e^{2x}$$

which won't work, as this is just $0 = 2e^{2x}$. Uh oh! What do we do? The problem here is that our guess is not linearly independent of our homogeneous solutions. What we do in this case is we just multiply our guess by x until we get a linearly independent guess.

In this case we would guess $y_p = Axe^{2x}$ and then solve for A to get $y_p = (1/2)xe^{2x}$, making the general form of our solution:

$$y = \frac{1}{2}xe^{2x} + c_1 e^{2x} + c_2 e^{-2x}.$$