Math 2280 - Lecture 8

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1 2nd-Order Linear Homogeneous Differential Equations with Constant Coefficients

We'll finish some of what we started last time. A second-order linear homogeneous differential equation is a differential equation of the form:

$$ay'' + by' + cy = 0$$

where $a, b, c \in \mathbb{R}$ are constants. Now, we know that if we can find two linearly independent solutions to the above equations, then we're done. In other words, for any given initial conditions, we can find the solution as a linear combination of our two linearly independent solutions. So, suppose we "guess" that our solution is of the form $y = e^{rx}$. If we plug this solution into our differential equation we get:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

Now, e^{rx} is never 0, and so we can divide both sides of this equation by it to get:

$$ar^2 + br + c = 0$$

So, if r is a root of the quadratic equation:

$$ax^2 + bx + c$$

then e^{rx} is a solution to the above differential equation.

Now, remembering back to junior high school, the roots of the quadratic equation are:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three possibilities, depending upon whether $b^2 - 4ac$, called the descriminant, is positive, negative, or 0.

If $b^2 - 4ac > 0$ then we get two distinct real roots, call them r_1, r_2. and so we get two linearly independent solutions, $e^{r_1 x}$ and $e^{r_2 x}$, and all of our solutions are of the form:

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

On the other hand, if $b^2 - 4ac = 0$ then we only have one (repeated) root, call it r, and our solutions are of the form:

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

Now, how do we know xe^{rx} is a solution? Well, try it out:

$$y'(x) = rxe^{rx} + e^{rx}$$
$$y''(x) = r^2 xe^{rx} + 2re^{rx}$$

and plugging these into our differential equation we get:

$$a(r^2 xe^{rx} + 2re^{rx}) + b(rxe^{rx} + e^{rx}) + cxe^{rx}$$
$$= xe^{rx}(ar^2 + br + c) + e^{rx}(2ar + b)$$

Now, the first term is 0 by definition, and if $b^2 - 4ac = 0$ then we have $r = -b/2a$, and so the second term is 0 as well. So, it's also a solution, and we have a second linearly independent solution.

Now, sorry to put this off some more, but we'll deal with the other case, $b^2 - 4ac < 0$, later.

2 General Solutions of Linear Equations

Last lecture we saw what happens when we have 2nd-order linear ODEs of the form:

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

Now, it only takes a little imagination to generalize to nth-order linear ODEs:

$$P_0(x)y^{(n)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x)$$

Now, if we assume $P_0(x) \neq 0$ on our interval of interest I then we can divide both sides by $P_0(x)$ to get:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

which has a corresponding homogeneous equation:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

As with the second-order case, for a homogeneous nth-order ODE we have any linear combination of solutions:

$$y = c_1y_1 + \cdots c_ky_k$$

is also a solution.

3 General Theory of nth-Order ODEs

Theorem - If $p_1, \ldots, p_n$ are continuous on I and $f(x)$ is too then:

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = f(x)$$

has a unique solution satisfying the initial conditions:

$$y(a) = b_0, y'(a) = b_1, \ldots, y^{(n-1)}(a) = b_{n-1}$$

for $b_k \in \mathbb{R}$ and $a \in I$.

Now, suppose we have a homogeneous nth-order ODE:

$$y^{(n)} + p_1 y^{(n-1)} + \cdots p_{n-1} y' + p_n y = 0$$

and we have n solutions $y_1, \ldots, y_n$. Can we get *all* solutions by solutions of the form:

$$y = c_1 y_1 + \cdots + c_n y_n?$$

Well, as with 2nd-order linear homogeneous ODEs the answer is yes, *if* the y_k are linearly independent. Again, as with 2nd-order linear homogeneous ODEs we can check for linear independence by using the Wronskian.

Definition
A set of functions $f_1, \ldots, f_n$ are linearly independent on an interval I provided:

$$c_1 f_1 + \cdots c_n f_n = 0$$

has no solutions on I except the trivial solution.

Now, how do we tell if they are linearly independent? As I said, we look at the Wronskian.

$$W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Now, again, the if everything is continuous on our interval of interest, then the Wronskian is either never 0 or always 0. If it's never 0 then the functions are linearly independent, if it's 0 then the functions are linearly dependent.

The general proof uses the uniqueness theorem in the same way as with 2nd-order linear homogeneous differential equations.

4 Nonhomogeneous Solutions

So far we've just been discussing homogeneous differential equations, but what happens if we have have a non-homogeneous differential equation? Well, suppose we have a non-homogeneous differential equation:

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y = f(x)$$

where $f(x) \neq 0$. Now, if we have a solution, which we'll for right now call y_{p_1}, and another solution which we'll call y_{p_2}, then if we take the difference of these two solutions $y_{p_1} - y_{p_2}$ the difference will solve the homogeneous equation:

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y = 0$$

and so will be of the form discussed above. In other words:

$$y_{p_2} = y_{p_1} + c_1 y_1 + c_2 y_2 + \cdots c_n y_n$$

where y_{p_1} is a particular solution to the given nonhomogeneous equation, and the y_i are n linearly independent solutions to the attendant homogeneous equation.

OK, so what does this mean? It means that if we can fine just one solution to the above nonhomogeneous equation, then we've found all the other solutions if we can solve the attendant homogeneous equation. Finding one solution can be hard, and finding the solutions of the attendant homogeneous equation can also be hard, but in many situations we'll find out it's possible and not *that* hard.

Example

Find the solution to the initial value problem:

$$y'' - 4y = 12$$
$$y(0) = 0,\ y'(0) = 10$$

Noting that a solution to the given differential equation is $y_p = -3$.

5 Homogeneous Equations with Constant Coefficients

So far we've larned how to solve 2nd-order homogeneous equations with constant coefficients, as long as the roots of the characteristic polynomial are not complex.

Well, we're now going to use this as a starting point for our study of more general equations, and we're going to figure out what to do if the roots are complex.

5.1 General Form of the Equation

The more general version of an nth order homogeneous linear differential equation with constant coefficients is:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

The first thing to notice is that if we try what we did in the second order case, namely plugging in the solution $y = e^{rx}$ and seeing if it works, we get:

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots a_1 r e^{rx} + a_0 e^{rx} = 0$$

Now, if we divide both sides by e^{rx}, which we know we can do for any value of x as e^{rx} is never 0, we get the equation:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

which is true if and only if r is a root of the *characteristic polynomial* for our differential equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

So, to repeat, if r is a root of the characteristic polynomial for our differential equation, then e^{rx} is a solution of our differential equation.

Now, the fundamental theorem of algebra says that for any polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

we can factor it as:

$$(x - c_1)^{r_1} (x - c_2)^{r_2} \cdots (x - c_m)^{r_m}$$

where $r_1 + r_2 + \cdots + r_m = n$, and the constants c_i could be complex numbers. This presents us with a number of possible situations.

5.2 Possible Situations

We can break down our possible situations into four distinct cases of varying complexity.

Case 1 - All roots of the characteristic polynomial are real and distinct.

If this is the case then we have n distinct solutions of the form $\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$, and all our solutions can be written in the form:

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

The Wronskian for these solutions, evaluated at 0, is:

$$W(0) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}.$$

This is something called the "Vandemonde determinant", and it's never 0 when the r_i are distinct. This is proven in one of the suggested homework problems.

Case 2 - Repeated Roots, all real.

Let's first just look at the case when our polynomial factors as:

$$(x - c)^n.$$

In this case our differential equation can be written as:

$$(D - c)^k y = 0$$

where D is the "differential operator", an operator that when applied to a function takes the derivative of that function. It's a linear operator, and if you raise it to any power, that just means you take that number of derivatives. For example:

$$(D-1)^2y = (D^2 - 2D + 1)y = y'' - 2y' + y$$

Now, *any* solution y to our differential equation can be written in the form $y(x) = u(x)e^{cx}$. If we apply the operator $(D - c)$ to this solution we get:

$$(D-c)u(x)e^{cx} = u'(x)e^{cx} + cu(x)e^{cx} - cu(x)e^{cx} = u'(x)e^{cx}.$$

If we repeat this process n times we get:

$$(D-c)u(x)e^{cx} = u^{(n)}(x)e^{cx}.$$

If $u(x)e^{cx}$ is a solution to our differential equation then we have $u^{(n)}(x)e^{cx} = 0$, which implies that $u^{(n)} = 0$, which means:

$$u(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1}$$

This give us a spanning set of independent functions $\{1, x, x^2, \ldots, x^{n-1}\}$, and a corresponding set of solutions to our ODE $\{e^{cx}, xe^{cx}, x^2e^{cx}, \ldots, x^{n-1}e^{cx}\}$.

We'll discuss the mixed case later.

Case 3 - Complex Roots.

Suppose that the characteristic polynomial for our polynomial is second order and can be factored as:

$$(x-r)(x-\overline{r})$$

where r is complex and $\overline{r}$ is its complex conjugate. Note that for any polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

if the a_i are real and if r is a complex root, then $\overline{r}$ must be a complex root as well. So, for any characteristic equation if we get a complex root, we'll get another root by taking its complex conjugate.

OK, so if r is a complex root, then we can write r as $r = a + bi$, and we get a solution to our differential equation:

$$e^{rx} = e^{(a+ib)x} = e^{ax} e^{ibx} = e^{ax}(\cos(bx) + i \sin(bx))$$

where we've used Euler's formula:

$$e^{i\theta} = \cos\theta + i \sin theta.$$

Now, the fact that $\overline{r} = a - ib$ is also a solution to our characteristic polynomial give us another solution:

$$e^{\overline{r}} = e^{ax}(\cos(bx) - i \sin(bx)).$$

A linear combination of these two can be written as:

$$k_1 e^{rx} + k_2 e^{\overline{r}x} = e^{ax}((k_1 + k_2)\cos(bx) + (k_1 - k_2) i \sin(bx)).$$

If we allow k_1, k_2 to be complex, then we can choose them so that the above equation is equal to:

$$e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$$

where the coefficients c_1 and c_2 are arbitrary *real* constants. So, we get two linearly independent solutions to our ODE $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$, and any other solution can be written as a linear combination of these two.

We'll incorporate the case of when you have repeated complex roots or multiple complex roots in our general case.

Case 4 - The General Case

Suppose we have a linear homogeneous ordinary differential equation with constant coefficients:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 = 0$$

To solve this problem, *in general*, what we do is we first calculate the roots of the characteristic polynomial:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (x - r_1)^{k_1}(x - r_2)^{k_2} \cdots (x - r_m)^{k_m}$$

Now, there will be some set of these roots, say r_1 through r_l, which are real. For each of these we will get a number of solutions equal to the degree of the root. So, for example, for our first root r_1 we will get solutions:

$$c_1 e^{r_1 x}, c_2 x e^{r_1 x}, \ldots, c_{k_1} x^{k_1 - 1} e^{r_1 x}$$

Now, as for the other roots, r_{l+1} through r_m, these will come in complex conjugate pairs, and each element of these pairs will have the same degree. So, for example, for the root r_{l+1} if we arrange out terms appropriately r_{l+2} will be its complex conjugate, and we'll have $k_{l+1} = k_{l+2}$. For this pair of roots we'll get the set of solutions:

$$c_1 e^{a_{l+1}x}\cos(b_{l+1}x), c_2 x e^{a_{l+1}x}\cos(b_{l+1}x), \ldots, c_{k_{l+1}} x^{k_{l+1}-1} e^{a_{l+1}x}\cos(b_{l+1}x)$$

and

$$d_1 e^{a_{l+1}x}\sin(b_{l+1}x), d_2 x e^{a_{l+1}x}\sin(b_{l+1}x), ldots, d_{k_{l+1}} x^{k_{l+1}-1} e^{a_{l+1}x}\sin(b_{l+1}x).$$

where the c_i and d_i are unknown constants, and the constants a_{l+1} and b_{l+1} are the real part and complex part, respectively, of the root $r_{l+1} = a_{l+1} + ib_{l+1}$.

Clear as mud? Yeah, in its full generality it's kind of confusing to state, but just know that this more general statement just builds up from the particular cases we've seen so far, and in this class and in most situations we'll only be dealing with relatively small order ODEs, so the full machinery of the more general statement won't be necessary in practice, but it's good to know.