Math 2280 - Lecture 7

Dylan Zwick

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1 The Runge-Kutta Method

So far we've learned two methods for approximating the solutions of an initial value problem:

$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$

Well, we're going to learn a third, and this one is much more powerful than Euler's method or the improved Euler's method. This is called the *Runge-Kutta* method.

Here's the idea behind the method. Suppose that we have computed the approximations $y_1, y_2, y_3, \ldots, y_n$ to the actual values $y(x_1), y(x_2), \ldots, y(x_n)$ and we want to approximate $y_{n+1} \approx y(x_{n+1})$. Well, then the fundamental theorem of calculus tells us:

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} y'(x)dx = \int_{x_n}^{x_n+h} y'(x)dx$$

Now, Simpson's rule for numerical integration tells us:

$$y(x_{n+1}) - y(x_n) \approx \frac{h}{6}\left[y'(x_n) + 4y'\left(x_n + \frac{h}{2}\right) + y'(x_{n+1})\right].$$

We can rewrite this as:

$$y_{n+1} \approx y_n + \frac{h}{6}\left[y'(x_n) + 2y'\left(x_n + \frac{h}{2}\right) + 2y'\left(x_n + \frac{h}{2}\right) + y'(x_{n+1})\right]$$

Now, we replace the slope value $y'(x_n)$ by:

$$y'(x_n) \approx f(x_n, y_n) = k_1$$

then, we use Euler's method to approximate one of the values of $y'\left(x_n + \frac{h}{2}\right)$:

$$y'\left(x_n + \frac{h}{2}\right) \approx f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) = k_2$$

then we use this slope value to approximate the other value of $y'\left(x_n + \frac{h}{2}\right)$:

$$y'\left(x_n + \frac{h}{2}\right) \approx f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) = k_3$$

and then we finally use this slope value to calculate our estimate for $y'(x_{n+1})$:

$$y'(x_{n+1}) \approx f\left(x_{n+1}, y_n + hk_3\right).$$

So, our final approximation is given by:

$$y(x_{n+1}) \approx y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right).$$

1.1 Error Bounds on Runge-Kutta

OK, that's a lot of work for each successive step. Is the extra work worth it? Well, yes, definitely. In fact, our maximum error is bounded by h^4, where h is the step size, in that:

$$|y(x_n) - y_n| \leq Ch^4$$

where C is a constant that depends upon the second derivative of $y(x)$, but does *not* depend on h. So, cutting our step size by 2 cuts down our maximum error by 16 times. In other words, it converges pretty quickly!

1.2 Example

Let's run through the Runge-Kutta method for two steps along our toy IVP:

$$\frac{dy}{dx} = y$$
$$y(0) = 1$$

For our first step we get:

$$k_1 = 1$$
$$k_2 = 1.25$$
$$k_3 = 1.3125$$
$$k_4 = 1.65625$$

So, our approximation for $y(.5)$ is:

$$y(.5) \approx y_1 = 1 + \frac{.5}{6}(1 + 2 \times 1.25 + 2 \times 1.3125 + 1.65625) = 1.6484375$$

Now, if we repeat this for one more step we get:

$$k_1 = 1.6484375$$
$$k_2 = 2.060543875$$
$$k_3 = 2.16357421875$$
$$k_4 = 2.73022460938$$

So, our approximation for $y(1)$ is:

$$y(1) \approx y_2 = y_1 + \frac{.5}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.71734619141 \approx 2.717.$$

I didn't write all the numbers out there, but you get the point. Anyways, our approximate value is very close to the actual value of e^1, and this is only using two steps. If we ran through this with ten steps (which I'm not going to do explicitly, thank you very much) we'd see that we have an *excellent* approximation for e^1.

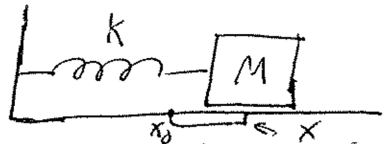
1.3 Final Remarks on These Methods

So, Runge-Kutta gives us a much better approximation, but each step or Runge-Kutta requires more computation, so what is it that we want? What we want is to be able to get an accurate estimate with a minimal amount of computation. So, while it may look like Runge-Kutta requires more computation, its advantage comes from us not having to use as small of step size to get an accurate approximation. So, although it looks more computationally intensive, what it turns out is that for most applications the Runge-Kutta method's value comes from it being able to provide us with an accurate approximation that requires fewer computations that Euler's method or the improved Euler's method. As paradoxical as it might sound, the Runge-Kutta method saves computation, and that's why it's useful.

2 Second Order Linear Equations

2.1 Initial Example

Suppose we take a mass m and attach it to a spring:



If we displace the mass a short distance x from its equilibrium there will be a restorative force F acting on it $F = -kx$, where k is the "spring constant". This is called Hooke's law. So, we have:

$$F = -kx$$

where

$$F = m\frac{d^2x}{dt^2}$$

where the second relation is Newton's second law. So, we have the relation:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

Now, one solution to this 2nd order ODE is:

$$x(t) = \sin\left(\sqrt{\frac{k}{m}}t\right)$$

another solution is

$$x(t) = \cos\left(\sqrt{\frac{k}{m}}t\right)$$

and, in fact, any linear combination:

$$x(t) = c_1 \sin\left(\sqrt{\tfrac{k}{m}}t\right) + c_2 \cos\left(\sqrt{\tfrac{k}{m}}t\right)$$

works as a solution. This raises some questions:

1. Does this cover *all* solutions to this ODE?
2. Does this handle *all* possible initial conditions of displacement and velocity?
3. Does this situation generalize?

The answer to all 3 of these questions is **yes**.

2.2 General Theory of 2nd Order Differential Equations

In general, we call a differential equation of the form:

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

a linear second order ODE. Note that the functions A, B, C, and F aren't necessarily linear.

If $F(x) = 0$ on the interval, I, of which we're interested then we call this a homogenous linear 2nd order ODE.

Our initial example was indeed a homogenous 2nd-order ODE with:

$$\begin{aligned} A(x) &= m \\ B(x) &= 0 \\ C(x) &= k \end{aligned}$$

Now, we saw that we can find two different functions that both solved the ODE, and in fact any linear combination of these functions also solved the ODE. This is true in general.

Theorem - For any homogenenous 2nd order ODE with solutions y_1, y_2 on I any function:

$$y = c_1 y_1 + c_2 y_2$$

is also a solution on I. This is pretty obvious and can be checked quite easily.

As for our other questions, just as in first order ODEs we have an existence and uniqueness theorem:

Theorem - Suppose that functions p, q and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0, b_1 the equation:

$$y'' + p(x)y' + q(x)y = f(x)$$

has a unique solution on all of I that satisfies:

$$y(a) = b_0, \, y'(a) = b_1.$$

So, the answers our second question.

2.3 Linear Independence of Two Functions

Now, as for our first and third questions before we answer these we need to lay some groundwork. First, a definition.

Definition - Two functions f, g defined on an open interval I are linearly independent on I provided that neither is a constant multiple of the other.

A pair of functions are linearly dependent if they're not linearly independent. (Well... duh!)

For two functions f, g we define the Wronskian:

$$W(x) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'.$$

Now, here's the important idea. If f, g are linearly dependent then

$$W(f, g) = 0 \text{ on } I.$$

On the other hand, if f, g are linearly independent then

$$W(f, g) \neq 0 \text{ on } \textit{every} \text{ point of } I.$$

Now, that *every* point fact is the important and amazing part.

Now, if y_1 and y_2 are linearly independent solutions of a 2nd order linear ODE. Then *all* solutions of the ODE are of the form:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

This can be proven without too much problem by using our existence and uniqueness theorem along with some linear algebra.

2.4 2nd Order Linear Homogeneous ODEs with Constant Coefficients

A linear homogeneous 2nd order ODE is an ODE of the form:

$$ay'' + by' + cy = 0$$

where a, b, c are constant.

If we try the solution $y(x) = e^{rx}$ then if we plug that in we get:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

Dividing through by e^{rx} we see that this solution works if r is a root of the quadratic equation:

$$ax^2 + bx + c = 0$$

So,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We'll only deal with distinct roots this time, but if the roots are distinct real numbers, then all our solutions are of the form:

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

where the roots are r_1 and r_2.

Example - What are the solutions of the differential equation:

$$y''(x) + 2y'(x) - 15y(x) = 0$$

Well, the roots of $x^2 + 2x - 15$ are $x = -5$ and $x = 3$. So, the solutions of our differential equation are:

$$y(x) = c_1e^{3x} + c_2e^{-5x}$$