Math 2280 - Lecture 5

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1 Equilibrium Solutions and Stability

1.1 Introduction

In the previous section we examined the simple population growth equation:

$$\frac{dx}{dt} = kx$$

where k is a constant. We also examined the more sophisticated logistic growth equation:

$$\frac{dx}{dt} = kx(M - x)$$

and saw that these equations were solved, respectively, by the solutions:

$$x(t) = x_0 e^{kx}$$

and

$$x(t) = \frac{Mx_0}{x_0 + (M - x_0)e^{kMt}}.$$

Now, in some sense we were lucky with these two equations, in that we were able to find explicit solutions without too much bother. Unfortuantely, this isn't always the case. However, even when it's difficult or impossible to solve a differential equation precisely, we can frequently still get important information about the behavior of the solutions by analyzing the form of the differential equaiton.

1.2 Phase Diagrams

A differential equation is called *autonomous* is it has the form:

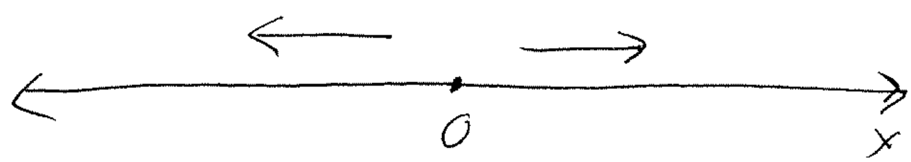
$$\frac{dx}{dt} = f(x)$$

This means that the differential equation does not depend explicitly on the independent variable t, although of course the variable x is a function of t.

Both of our population growth equations are autonomous differential equations. Now, for each of these we can draw something called a phase diagram:

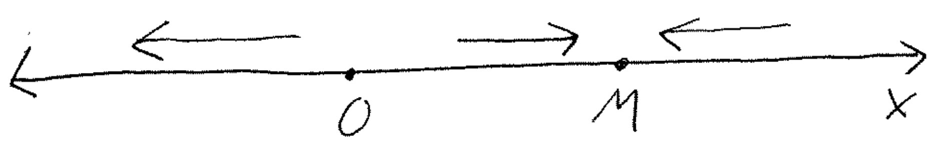
$$\frac{dx}{dt} = kx$$

Phase Diagram



$$\frac{dx}{dt} = kx(M - x)$$

Phase Diagram



Now, what we do to create these phase diagrams is that we solve for the *critical points* of the function $f(x)$. These critical points are the points where the function $f(x) = 0$. Now, in between these critical points, if we assume (as we will) that $f(x)$ is continuous, the function will be either positive or negative.

Now, we draw out a portion of the $x - axis$ containing all the critical points, and we mark the critical points with dots. Then, above the segments in between these critical points we draw a left arrow if $f(x)$ is negative on the segment, and a right arrow if $f(x)$ is positive on the segment. We also draw the appropriate arrows for the region greater than any critical point and less than any critical point.

These critical points represent what are called *equilibrium solutions* to our differential equation. These are solutions of the form $x(t) = c$, where c is a constant.

1.3 Stability of Critical Points

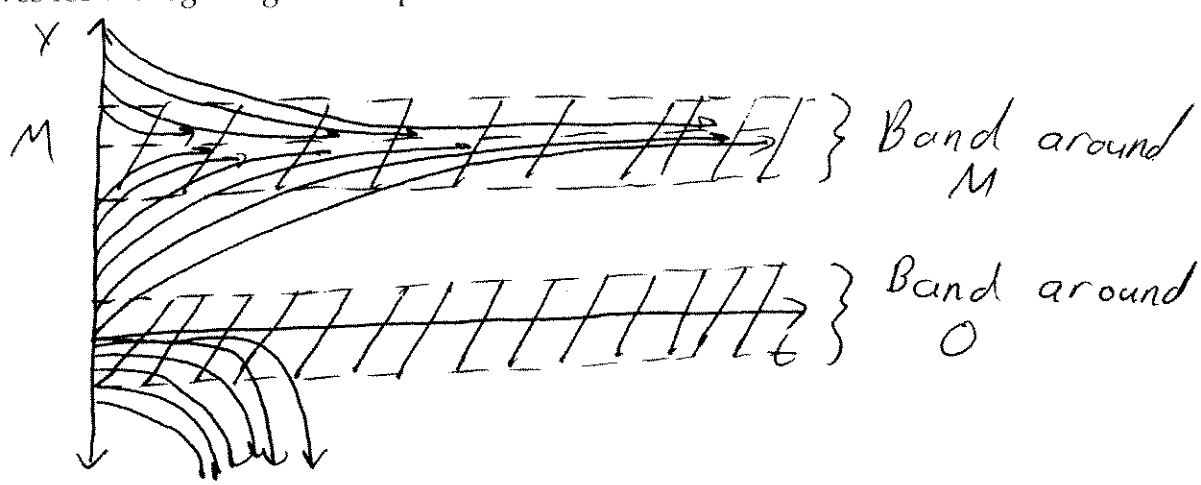
The technical definition of stability of a critical point is this:

Definition - The critical point $x = c$ is *stable* if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$|x_0 - c| < \delta \text{ implies that } |x(t) - c| < \epsilon$$

Now, what this is saying is that if you start our sufficiently close to the critical point, within some "band" around the critical point, that you'll always stay within that band.

We can see this phenomenon in action if we look at some solution curves for the logistic growth equation:



We can see that for the critical point $x = M$ we have a stable critical point, and that solutions around the point "funnel" in towards it. The critical point $x = 0$ on the other hand is an unstable critical point, and we can see that solutions close to it diverge.

Now, it's easy to tell from a phase diagram which critical points are stable and which are not. If your critical point has two arrows going into it, then it's stable. If it has two arrows going away from it, then it's unstable. There can also exist the (rare) situation where a critical point has one arrow going into it and one arrow going out of it. Such a situation we call semistable.

1.4 Harvesting a Logistic Population

The autonomouos differential equation:

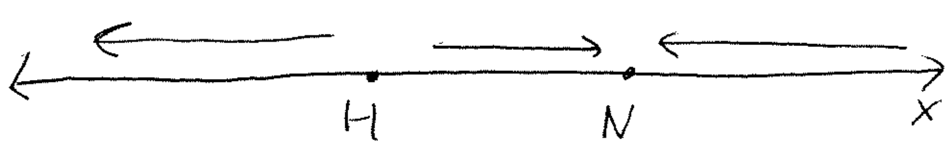
$$\frac{dx}{dt} = kx(M - x) - h$$

may be considered to describe a logistic population with harvesting. For instance, we might think of the population of fish in a lake from which h first per year are removed by fishing.

If we solve for the critical points of this differential equation, the quadratic equation tells us these critical points are:

$$c = \frac{kM \pm \sqrt{(kM)^2 - 4hk}}{2k}$$

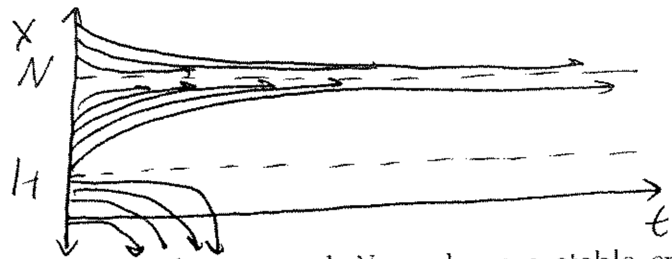
Now, if $h < \frac{kM^2}{4}$ then we will have two solutions, call them H and N, where $H < N$. In this case we'll have a phase diagram that looks like:



and our solution (which you can check on your own) will be:

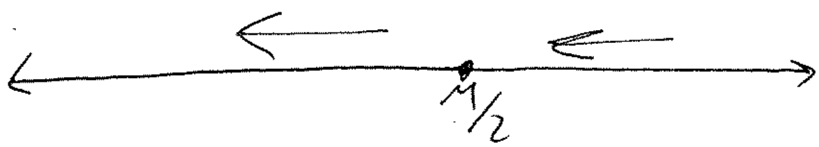
$$x(t) = \frac{N(x_0 - H) - H(x_0 - N)e^{-k(N-H)t}}{(x_0 - H) - (x_0 - N)e^{-k(N-H)t}}.$$

Now, if we graph some representative solution curves of this differential equation we'll get a picture that looks like:

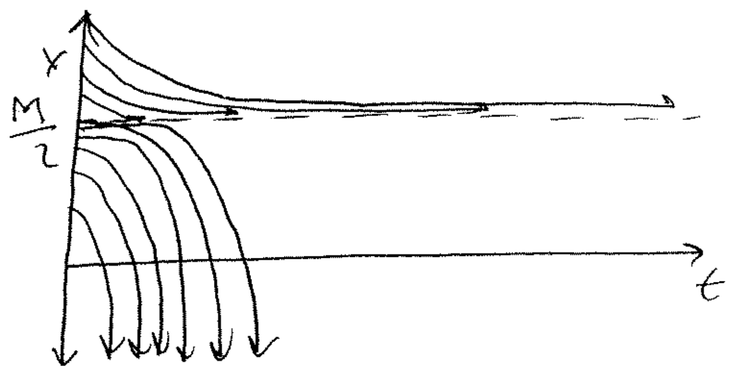


and we can see that around N we have a stable critical point, and around H we have an unstable critical point. What this means is that for any initial value greater than H our population size will approach N as time goes on. For any initial value less than H our population size will approach $-\infty$ in a (finite!) amount of time. Of course, in our physical model, we'd say you can't have less than 0 fish, and so the model would definitely break down when your population became negative.

Now, if $h = \frac{kM^2}{4}$ then we'd have a situation with just one critical point $M/2$, and a phase diagram tht looks like this:

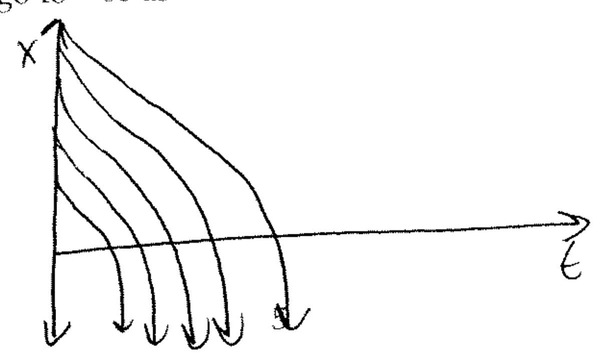


Here our solution curves would look like this:



and we'd have what's called a semistable equilibrium.

For $h > \frac{kM^2}{4}$ we would have no critical points, and no matter what our solutions would go to $-\infty$ as time increased. We'd have solutions curves that look like:

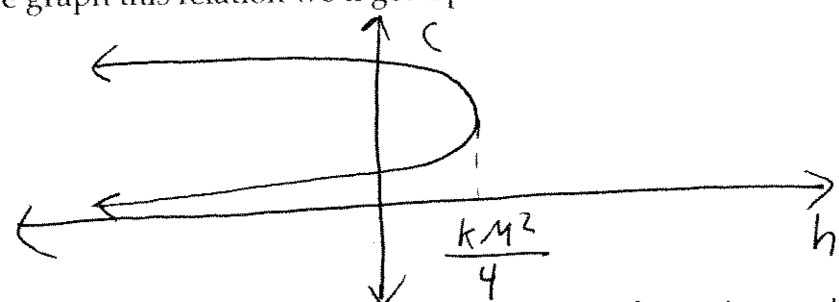


1.5 Bifurcation

We can actually see that there's a relation between our critical points and the value of our initial paramater h. The relation can be written as:

$$h = k(Mc - c^2).$$

If we graph this relation we'll get a parabolic curve of the form below:



This is called a bifurcation diagram. It tells us for a given value of h how many critical points we have, and what these critical points will be. We may have much more to say about these bifurcation diagrams in later chapters.

2 Acceleration-Velocity Models

If an object is close to the earth's surface and we neglect any effects of air resistance, then the object will experience a constant downward force from gravity, and Newton's second law tells us that:

$$m\frac{dv}{dt} = F_G$$

where m is the object's mass, v is the object's velocity, and F_G is the constant force from gravity, which will be $-mg$.

Now, this is a very simple, and can be an OK model for some very simple physical situations, but in almost every situation in real life, even pretty simple ones, we'll need to take air resistance into account. Now, the phenomenon of air resistance is a pretty complicated one, so we'll just take a look at a relatively simple model that incorporates an approximation to air resistance that works pretty well in many situations.

2.1 Resistance Proportional to Velocity

The first we'll consider is the situation where air resistance is proportional to velocity, and in the direction opposite the direction of the velocity:

$$F_R = -kv$$

where k is a positive constant, and v is the object's velocity. Now, combining this air resistance with the (still assumed to be constant) force from gravity and using Newton's second law we get:

$$m\frac{dv}{dt} = -kv - mg$$
$$\Rightarrow \frac{dv}{dt} = -\rho v - g$$

where $\rho = \frac{k}{m}$.
Now, if we solve this differential separable differential equation we get:

$$v(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}.$$

Now, we note that as $t \to \infty$ our velocity approaches the value $-\frac{g}{\rho}$. This is called the object's *terminal velocity*. The absolute value of this is called the object's *terminal speed* and is given by:

$$|v_\tau| = \frac{mg}{k}.$$

This phenomenon of terminal speed is what makes skydiving possible. Now, the textbook also covers an example where the force of air resistance is proportional to the square of the velocity. No new concepts in the analysis, but I'd recommend reading through it, although we won't go over it in lecture.

2.2 Variable Gravitational Acceleration

Now, the model of constant gravitation only works when we're close to the surface of the earth, and the distances we're dealing with are small relative to the radius of the earth. If we start to deal with larger distances, then we must take into account that the acceleration from gravity is weaker the farther we are away from the earth. Newton's law of universal gravitation tells us that the force from gravity experienced a distance r from the center of the earth will be:

$$F = \frac{GmM}{r^2}$$

where m is the mass of the object, M is the mass of the earth, G is Newton's gravitational constant $G = 6.67 \times 10^{-11} Nm/kg^2$.

We can use this relation to calculate an object's escape velocity on the surface of the earth. This is the speed at which an object must be moving away from the earth at the earth's surface if it is to break free from the gravitational attraction of the earth and continue to move away "forever".

Well, we note that if we move away from the earth along a line that goes through the earth's center, then Newton's second law tells us:

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}.$$

If we note that $v = \frac{dr}{dt}$ then we can transform this relation into:

$$v\frac{dv}{dr} = -\frac{GM}{r^2}.$$

If we integrate both sides with respect to r we get:

$$\frac{1}{2}v^2 = \frac{GM}{r} + C.$$

If we say $v(0) = v_0$ and solve this for C we get:

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right).$$

Now, if the object is to escape from the "clutch" of the earth then its velocity must always be positive as $r \to \infty$. This is possible if

$$v_0 \geq \sqrt{\frac{2GM}{R}}.$$

So, the escape velocity for the earth (or for any planet of given mass M) is:

$$v_0 = \sqrt{\frac{2GM}{R}}.$$

For the earth the escape velocity is $v_0 \approx 11,180 m/s$.