

MATH 2280 - LECTURE 25

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1. EXAMPLE PROBLEMS

So, chapters 8 and 9 involve a lot of very difficult problems. For our last lecture, we'll go over a few of these problems and how you do them. So, no new material, just some examples.

1.1. Series Solutions Near Ordinary Points. Find the general solution in powers of x of the differential equation:

$$y'' - x^2y' - 3xy = 0.$$

Solution

We first note that as the coefficient in front of y'' is a constant, and the coefficients in front of the lower order terms are polynomials, that $x = 0$ is an ordinary point of the differential equation.

So, we assume that we can find a solution in the form:

$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} c_n x^n \\y'(x) &= \sum_{n=1}^{\infty} n c_n x^{n-1} \\y''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.\end{aligned}$$

Now, if we plug these values into our differential equation we get the relation:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} - 3x \sum_{n=0}^{\infty} c_n x^n = 0.$$

Multiplying through we get:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^{n+1} - \sum_{n=0}^{\infty} 3c_n x^{n+1} = 0$$

Now, the only place we're going to have an x^0 term is in the first sum, when $n = 2$, and so given our constant term must be 0 this implies $c_2 = 0$.

For higher order terms all of our sums are going to enter into the calculation, and if we shift our index on the first one by 3, and note that starting the second series at 0 is the same as starting it at 1, we get:

$$\sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} - n c_n - 3c_n] x^{n+1} = 0.$$

The coefficient must be 0 for all n , and so solving this recursion relation we get:

$$c_{n+3} = \frac{c_n}{n+2}.$$

We note that this requirement places no restriction on the terms c_0 and c_1 , so these terms are arbitrary, and represent the arbitrary constants in our general solution.

Now, in some sense we're done, in that we've figured out how to represent every term in our power series in terms of our two arbitrary constants c_0 and c_1 . However, if possible we'd like to derive a closed form solution for our terms.

If we take a look at our first few terms we get:

$$\begin{aligned} c_3 &= \frac{c_0}{2}, \\ c_4 &= \frac{c_1}{3}, \\ c_5 &= \frac{c_2}{4} = 0, \\ c_6 &= \frac{c_3}{5} = \frac{c_0}{5 \times 2}, \\ c_7 &= \frac{c_4}{6} = \frac{c_1}{6 \times 3}, \\ c_8 &= \frac{c_5}{7} = 0, \end{aligned}$$

$$\begin{aligned}
c_9 &= \frac{c_6}{8} = \frac{c_0}{8 \times 5 \times 2}, \\
c_{10} &= \frac{c_7}{9} = \frac{c_1}{9 \times 6 \times 3}, \\
&\vdots
\end{aligned}$$

From these first few terms we can already see our pattern emerging. This pattern is:

$$\begin{aligned}
c_{3n} &= \frac{c_0}{(3n-1) \times (3n-4) \times \cdots \times 8 \times 5 \times 2} \text{ for } n > 0, \\
c_{3n+1} &= \frac{c_1}{3n \times (3n-3) \times \cdots \times 9 \times 6 \times 3} = \frac{c_1}{3^n n!}, \\
c_{3n+2} &= 0.
\end{aligned}$$

So, our closed form solution is:

$$y(x) = c_0 \left(1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(3n-1) \times (3n-4) \times \cdots \times 8 \times 5 \times 2} \right) + c_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{3^n n!}.$$

1.2. Series Solutions Near Regular Singular Points. Find the general solution to the differential equation:

$$2xy'' - y' - y = 0.$$

Solution

First we note that if we divide everything through by $2x$ the coefficients in front of both y' and y are not analytic at $x = 0$. However, $p(x) = xP(x) = -1/2$ and $q(x) = x^2Q(x) = -x/2$ both are, so $x = 0$ is a regular singular point.

Now, the constant term of $p(x)$ is $-1/2$, while the constant term of $q(x)$ is 0. So, the indicial equation is:

$$r(r-1) - \frac{r}{2} = 0,$$

which has solutions $r = 0$ and $r = 3/2$. As the difference between these two solutions is not an integer, we know we'll be able to find two Frobenius solutions. So, our solutions will be of the form:

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r},$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1},$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}.$$

Plugging these into our differential equation we get:

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Now, the lowest order term is x^{r-1} , and the coefficient there is:

$$[r(r-1) - r/2]c_0.$$

Now, for our two values of r this is automatically satisfied (it's just the indicial equations, which our two values of r solve by definition), and so c_0 is arbitrary.

For the higher order terms, simplifying the above series we get:

$$\sum_{n=1}^{\infty} [(2(n+r+1)(n+r) - (n+r+1))c_{n+1} - c_n] x^{n+r} = 0.$$

Now, each of these coefficients must be 0, and so we get the recursion relation:

$$c_{n+1} = \frac{c_n}{2(n+r+1)(n+r) - (n+r+1)}.$$

Now, if we plug in our values of r we get for $r = 0$:

$$c_{n+1} = \frac{c_n}{2(n+1)n - (n+1)} = \frac{c_n}{(2n-1)(n+1)}.$$

We note that for no integer value of $n \geq 1$ is the denominator 0, so this defines our series for all values of n for which we're interested.

On the other hand, if we plug in $r = 3/2$ we get:

$$c_{n+1} = \frac{c_n}{(n+1)(2n+5)}.$$

We again note that for no integer value of $n \geq 1$ is the denominator 0, so this again defines our series for all n in which we're interested.

Now, as mentioned earlier, we could say we're done here. However, let's see if we can find a closed form solution for our series.

if we examine the $r = 0$ terms we get:

$$\begin{aligned} c_1 &= -\frac{c_0}{1}, \\ c_2 &= -\frac{c_1}{2} = -\frac{c_0}{2}, \\ c_3 &= -\frac{c_2}{3 \times 3} = -\frac{c_0}{(3 \times 1) \times (3 \times 2 \times 1)}, \\ c_4 &= -\frac{c_3}{5 \times 4} = -\frac{c_0}{(5 \times 3 \times 1) \times (4 \times 3 \times 2 \times 1)}, \\ &\vdots \end{aligned}$$

We can see the pattern here, and so we get:

$$\begin{aligned} c_n &= -\frac{c_0}{((2(n-1)-1) \times (2(n-1)-3) \times \cdots \times 3 \times 1) \times (n \times (n-1) \times \cdots \times 3 \times 2 \times 1)} \\ &= -\frac{c_0 2^{n-1} (n-1)!}{(2(n-1))! n!} = -\frac{c_0 2^{n-1}}{n \times (2(n-1))!}. \end{aligned}$$

So, one of our Frobenius solutions is:

$$y_1(x) = c_0 \left(1 - \sum_{n=1}^{\infty} \frac{x^n 2^{n-1}}{n \times (2(n-1))!} \right).$$

On the other hand, for our $r = 3/2$ series we get:

$$\begin{aligned} c_1 &= \frac{c_0}{1 \times 5}, \\ c_2 &= \frac{c_1}{2 \times 7} = \frac{c_0}{(2 \times 1) \times (7 \times 5)} = \frac{3c_0}{(2 \times 1) \times (7 \times 5 \times 3 \times 1)}, \\ c_3 &= \frac{c_2}{3 \times 9} = \frac{3c_0}{(3 \times 2 \times 1) \times (9 \times 7 \times 5 \times 3 \times 1)}, \\ &\vdots \end{aligned}$$

We can already see where this is going. The denominator in each term is just $n!$ multiplied by the product of the first $n + 2$ odd terms. Writing this out in closed form we get:

$$c_n = \frac{3c_0}{\left(\frac{(2(n+2))!}{2^{n+2}(n+2)!}\right) n!} = \frac{3(n+2)(n+1)2^{n+2}c_0}{(2(n+2))!}.$$

So, our second Frobenius solution is:

$$y_2(x) = c_0 x^{\frac{3}{2}} \left(1 + \sum_{n=1}^{\infty} \frac{3(n+2)(n+1)2^{n+2}x^n}{(2(n+2))!} \right).$$

So, when the smoke clears, our solution is:

$$y(x) = c_0 \left(1 - \sum_{n=1}^{\infty} \frac{x^n 2^{n-1}}{n \times (2(n-1))!} \right) + c_1 x^{\frac{3}{2}} \left(1 + \sum_{n=1}^{\infty} \frac{3(n+2)(n+1)2^{n+2}x^n}{(2(n+2))!} \right).$$

Where we've renamed our second constant c_1 so as to not give it the same symbol as our first constant.

1.3. Calculating Fourier Series. Calculate the Fourier series for the periodic function $f(t)$ where one period is given by:

$$f(t) = t, \quad -2 < t < 2.$$

Solution

This function is periodic with period 4, and $4/2 = 2$. So, the Fourier series for this function will be:

$$\frac{a_0}{2} = \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{n\pi t}{2} \right) + b_n \sin \left(\frac{n\pi t}{2} \right) \right),$$

where the constants are given by:

$$a_n = \frac{1}{2} \int_{-2}^2 t \cos \left(\frac{n\pi t}{2} \right) dt;$$

$$b_n = \frac{1}{2} \int_{-2}^2 t \sin \left(\frac{n\pi t}{2} \right) dt.$$

For any of the a_n terms we note that the integrand is an odd function, and so the integral will be 0. As for the b_n we get:

$$b_n = \frac{1}{2} \int_{-2}^2 t \sin\left(\frac{n\pi t}{2}\right) dt = \int_0^2 t \sin\left(\frac{n\pi t}{2}\right) dt.$$

Integrating these by parts we get:

$$\int t \sin\left(\frac{n\pi t}{2}\right) dt = \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi t}{2}\right) - \frac{2}{n\pi} t \cos\left(\frac{n\pi t}{2}\right).$$

If we evaluate this at the appropriate limits we get:

$$b_n = \int_0^2 t \sin\left(\frac{n\pi t}{2}\right) dt = -\frac{4}{n\pi}(-1)^n = \frac{4(-1)^{n+1}}{n\pi}.$$

So, our Fourier series will be:

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi t}{2}\right). \\ &= \frac{4}{\pi} \left(\sin\left(\frac{\pi t}{2}\right) - \frac{1}{2} \sin\left(\frac{2\pi t}{2}\right) + \frac{1}{3} \sin\left(\frac{3\pi t}{2}\right) - \dots \right). \end{aligned}$$