MATH 2280 - LECTURE 23

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1. The Concept of Fourier Series

The last subject we'll deal with in this class is Fourier series, which are some of the most interesting and most useful objects (or methods, or whatever) in mathematics. Fourier series are used all the time for both practical and theoretical mathematics, and there are whole advanced, and very advanced, classes on the subject. So, needless to say, in the last few lectures we'll only be putting our toe in the ocean. However, we can learn enough to get some useful and interesting results, and to understand the basic idea behind the method.

So, without further ado, let's begin by taking a look at a relatively simple differential equation that we've encountered before:

$$x''(t) + \omega_0^2 x(t) = f(t).$$

Now, we've learned how to solve this ODE for a number of possible functions $f(t)$. We know the solution for $f(t) = 0$, or when $f(t)$ is made up of sums and products of exponentials, polynomials, sines, and cosines. As a specific example, suppose:

$$f(t) = A\cos(\omega t).$$

Then we learned long ago that a particular solution to this ODE is:

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t)$$

as long as $\omega_0 \neq \omega$. Using the linearity of our differential equation (a property which becomes incredibly important when dealing with Fourier series) we can induce from this solution that if:

$$f(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t),\ \omega_n \neq \omega_0 \text{ for any } n.$$

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then our particular solution will be:

$$f(t) = \sum_{n=1}^{N} \frac{A_n}{\omega_0^2 - \omega_n^2} \cos(\omega_n t).$$

Nothing new here, but this observation is the starting point for the study of Fourier series. What is says is that for any function $f(t)$ if it can be represented as a sum of cosine functions, then we know how to solve it. This idea can be immediately extended to functions that can be represented as sums of sine and cosine functions.

1.1. **Periodic Functions.** A function is periodic with period p if there exists a number $p > 0$ such that:

$$f(t+p) = f(t) \text{ for all } t.$$

The smallest such p, if a smallest one exists, is called *the* period (also sometime the *fundamental period*) of the function.

For any piecewise continuous function $f(t)$ of period 2π we can define its *Fourier series*:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt)).$$

The coefficients of this series are defined by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

for $n = 0, 1, 2, \ldots$, and:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

for $n = 1, 2, \ldots$.

1.2. **Calculating Fourier Transforms.** Now, we first note a few facts about integrals of this kind.

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases};$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases};$$

$$\int_{-\pi}^{\pi} \cos(mt)\sin(nt)dt = 0 \text{ always.}$$

Proving these just involves some clever use of trigonometric identities. Here's how you'd prove the second relation.

Proof

We first note the trigonometric identities:

$$\cos((m+n)t) = \cos(mt)\cos(nt) - \sin(mt)\sin(nt)$$

$$\text{and}$$

$$\cos((m-n)t) = \cos(mt)\cos(nt) + \sin(mt)\sin(nt).$$

Using these relations we get that:

$$\sin(nt)\sin(mt) = \frac{\cos((m-n)t) - \cos((m+n)t)}{2}.$$

Therefore, our integral becomes:

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \int_{-\pi}^{\pi} \frac{\cos((m-n)t) - \cos((m+n)t)}{2}dt.$$

Now, if $m \neq n$ this integral evaluates to:

$$\frac{1}{2}\left[\frac{\sin((m-n)t)}{(m-n)} - \frac{\sin((m+n)t)}{(m+n)}\right]|_{-\pi}^{\pi}$$

$$= \frac{1}{2}(0-0) - \frac{1}{2}(0-0) = 0.$$

On the other hand if $m = n$ then our integral is just:

$$\int_{-\pi}^{\pi} \frac{1}{2}dt = \pi.$$

Example - Find the Fourier series of the square-wave function:

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \\ 0 & t = \{-\pi, 0, \pi\} \end{cases}$$

Solution - The Fourier series is:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_{-\pi}^{0} dt + \frac{1}{\pi}\int_{0}^{\pi} dt$$

$$= -\frac{1}{\pi}(\pi) + \frac{1}{\pi}(\pi) = -1 + 1 = 0.$$

$$\text{for } n > 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \frac{1}{\pi}\int_{-\pi}^{0}(-cos(nt))dt + \frac{1}{\pi}\int_{0}^{\pi}\cos(nt)dt$$

$$= -\frac{\sin(nt)}{n\pi}|_{-\pi}^{0} + \frac{\sin(nt)}{n\pi}|_{0}^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = -\frac{1}{\pi}\int_{-\pi}^{0}\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi}\sin(nt)dt$$

$$= -\frac{1}{\pi}\left(\frac{-\cos(nt)}{n}\right)|_{-\pi}^{0} - \frac{1}{\pi}\left(\frac{\cos(nt)}{n}\right)|_{0}^{\pi} = \frac{2}{n\pi}(1-\cos(n\pi))$$

$$= \frac{2}{n\pi}[1-(-1)^n]$$

$$\text{and so}$$

$$b_n = \begin{cases} 0 & even \\ \frac{4}{n\pi} & odd \end{cases}$$

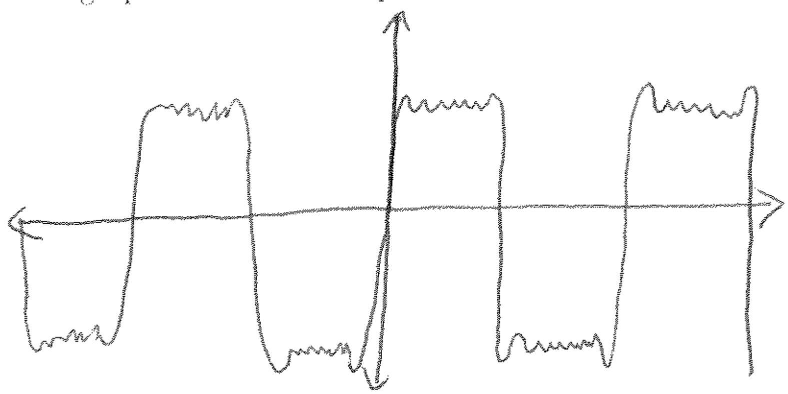
Therefore, the Fourier transform of the function $f(t)$ is:

$$f(t) \sim \frac{4}{\pi}\sum_{odd}\frac{\sin(nt)}{n}.$$

The partial sums of this series are:

$$S_n(t) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{\sin((2n-1)t)}{2n-1}.$$

The graph of one of these partial sums looks like:



Example
Find the Fourier transform of the function:

$$f(t) = \begin{cases} 3 & -\pi < t \leq 0 \\ -2 & 0 < t \leq \pi \end{cases}$$

Solution
The coefficients of this Fourier series will be:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}[3\pi - 2\pi] = 1;$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 3\cos(nt)dt - \int_{0}^{\pi} 2\cos(nt)dt\right]$$

$$= -\frac{3}{n\pi}\sin(nt)|_{-\pi}^{0} - \frac{2}{n\pi}\sin(nt)|_0^{\pi} = 0;$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 3\sin(nt)dt - \int_{0}^{\pi} 2\sin(nt)dt\right]$$

$$= -\frac{3}{n\pi}\cos(nt)|_{-\pi}^{0} - \frac{2}{n\pi}\cos(nt)|_0^{\pi}$$

$$= -\frac{3}{n\pi} + \frac{3}{n\pi}(-1)^n + \frac{2}{n\pi}((-1)^n - 1)$$

so

$$b_n = \begin{cases} 0 & even \\ -\frac{10}{n\pi} & odd \end{cases}$$

Therefore, the Fourier transform of $f(t)$ is:

$$f(t) \sim \frac{1}{2} - \frac{10}{\pi}\left[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right].$$

1.3. **General Fourier Series.** Suppose we have a function $f(t)$ of period $2L$, where $L > 0$. Then if we define:

$$g(u) = f\left(\frac{Lu}{\pi}\right),$$

we see $g(u)$ is 2π periodic. So, the Fourier transform of $g(u)$ is:

$$g(u) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos{(nu)} + b_n \sin{(nu)}.$$

The coefficients are:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos{(nu)}du,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin{(nu)}du.$$

If we then define $t = \frac{Lu}{\pi}$ then $f(t) = g\left(\frac{\pi t}{L}\right)$. Therefore, we get the corresponding "Fourier series" for $f(t)$:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt.$$

So, this defines a more general concept of a Fourier series of period $2L$. We note that taking the above integrals from $-L$ to L is, given the periodicity of the function, completely arbitrary, and we could integrate instead over *any* interval of length $2L$.

1.4. **Convergence of Fourier Series. The Convergence Theorem** - Suppose that the periodic function $f(t)$ is piecewise smooth. Then its Fourier series converges to:

(1) the value $f(t)$ at each point where f is continuous
(2) the value $\frac{1}{2}[f(t^+) + f(t^-)]$ at each point where $f(t)$ is discontinuous.

Example
Find the Fourier series of a square wave function of period 4.

Solution - Using our earlier result about the square wave of period 2π we get the Fourier transform:

$$f(t) \sim \frac{4}{\pi}\left(\sin\left(\frac{\pi t}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{2}\right) + \cdots\right).$$

We note that if we plug in $t = 1$ and use the above convergence theorem we get that $f(1) = 1$ and the relation:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right),$$

which is the famous Leibniz formula.