# MATH 2280-LECTURE 22 

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## 1. Regular Singular Points

Last time we looked at how to solve linear ODEs of the form:

$$
A(x) y^{\prime \prime}+B(x) y^{\prime}+C(x) y=0
$$

The first thing we do is rewrite the ODE as:

$$
\begin{gathered}
y^{\prime \prime}+P(x) y^{\prime}+Q(x) y=0 \\
\text { where, of course }, \\
P(x)=\frac{B(x)}{A(x)}, \text { and } Q(x)=\frac{C(x)}{A(x)} .
\end{gathered}
$$

Now, if $P(x)$ and $Q(x)$ are analytic around the point $a$ then we have two linearly independent solutions of the form:

$$
y(x)=\sum_{n=0}^{\infty} c_{n}(x-a)^{n}
$$

where the raddi of convergence are at least as great as the distance in the complex plane from $a$ to the nearest singular point of either $P(x)$ or $Q(x)$.
1.1. Ordinary, Regular, and Irregular Points. We first state without proof the fact that either $P(x)$ and $Q(x)$ are analytic at $x=a$ of approach $\pm \infty$ as $x \rightarrow a$.

Now, of course, we must ask the question of what we do if either $P(x)$ or $Q(x)$ is not analytic at $a$ ? Well, it turns out we have methods for dealing with this if they're not analytic in the "right way". We'll also restrict ourselves to dealing with the case $a=0$, but we note that by just shifting our coordinates this restriction incurs no loss of generality.

[^0]We divide singular points into two types: regular singular points, and irregular singular points. A regular singular point is a singular point where, if we rewrite:

$$
\begin{aligned}
& y^{\prime \prime}+P(x) y^{\prime}+Q(x) y=0 \\
& \text { as } \\
& y^{\prime \prime}+\frac{p(x)}{x} y^{\prime}+\frac{q(x)}{x^{2}} y=0
\end{aligned}
$$

the functions $p(x)$ and $q(x)$ are analytic. We will go over how to solve second order linear ODEs around regular singular points using series methods. We will not discuss how to solve ODEs around irregular singular points, as that is a much more difficult and advanced topic.

Example - Determine whether $x=0$ is an ordinary point, a regular singular point, or an irregular singular point of the ODE:

$$
x^{2} y^{\prime \prime}+(6 \sin x) y^{\prime}+6 y=0
$$

Solution - If we divide through by $x^{2}$ we get:

$$
y^{\prime \prime}+\frac{6 \sin x}{x^{2}} y^{\prime}+\frac{6}{x^{2}} y=0
$$

which has $\lim _{x \rightarrow 0}=\infty$ for both coefficient functions, so it's not an ordinary point. However, if we rewrite this in the format above, we see:

$$
p(x)=\frac{6 \sin x}{x}, q(x)=6,
$$

both of which are analytic at $x=0$. So, $x=0$ is a regular singular point of the ODE.

We again state a fact without proof, this time that if the limits:

$$
\lim _{x \rightarrow 0} p(x) \text { and } \lim _{x \rightarrow 0} q(x)
$$

exist, are finite, and are not 0 then $x=0$ is a regular singular point. If both limits are 0 then $x=0$ may be a regular singular point or an ordinary point. If either limit fails to exists or is $\pm \infty$ then $x=0$ is an irregular singular point. So, this gives us a useful way of testing if a singular point is regular.
1.2. The Method of Frobenius. Now we'll figure out how to actually solve these ODEs around regular singular points. We start by examining the simplest such ODE:

$$
x^{2} y^{\prime \prime}+p_{0} x y^{\prime}+q_{0} y=0
$$

where $p_{0}, q_{0}$ are both constants. This ODE is solved by $y=x^{r}$, where $r$ satisfies the quadratic:

$$
r(r-1)+p_{0} r+q_{0}=0 .
$$

Using this as our starting point, in general we assume our solution has the form:

$$
y(x)=x^{r} \sum_{n=0}^{\infty} c_{n} x^{n} .
$$

Note - This is NOT a power series if $r \notin \mathbb{Z}^{+}$.
This is called a Frobenius series. Now, we want to take a look at what this constant $r$ needs to be and at what kind of series we get. So, assume that we have a solution in this form. In this case we have:

$$
\begin{gathered}
y(x)=x^{r} \sum_{n=0}^{\infty} c_{n} x^{n}=\sum_{n=0}^{\infty} c_{n} x^{n+r}, \\
y^{\prime}(x)=\sum_{n=0}^{\infty} c_{n}(n+r) x^{n+r-1}, \\
y^{\prime \prime}(x)=\sum_{n=0}^{\infty} c_{n}(n+r)(n+r-1) x^{n+r-2} .
\end{gathered}
$$

If we substitute this into:

$$
x^{2} y^{\prime \prime}+x p(x) y^{\prime}+q(x) y=0
$$

where $p(x)$ and $q(x)$ are analytic around $x=0$ and so have a power series representation of the form:

$$
\begin{aligned}
p(x) & =p_{0}+p_{1} x+p_{2} x^{2}+\cdots \\
q(x) & =q_{0}+q_{1} x+q_{2} x^{2}+\cdots
\end{aligned}
$$

then plugging everything in we get:

$$
\begin{gathered}
{\left[r(r-1) c_{0} x^{r}+(r+1) r c_{1} x^{r+1}+\cdots\right]+\left[p_{0} x+p_{1} x^{2}+\cdots\right] \cdot\left[r c_{0} x^{r-1}+\right.} \\
\left.(r+1) c_{1} x^{r}+\cdots\right]+\left[q_{0}+q_{1} x+\cdots\right] \cdot\left[c_{0} x^{r}+c_{1} x^{r+1}+\cdots\right]=0 .
\end{gathered}
$$

Now, if we look at the $x^{r}$ term we get, assuming (as we of course always can and should) that $c_{0} \neq 0$, we get the relation:

$$
r(r-1)+p_{0} r+q_{0}=0
$$

This is called the indicial equation of the ODE, and it must, according to the identity principle, be satisfies for our solution to work. This is, of course, only a necessary condition, and we certainly haven't proven it's sufficient. That's where the next theorem comes in:

Theorem - Suppose that $x=0$ is a regular singular point of the ODE:

$$
x^{2} y^{\prime \prime}+x p(x) y^{\prime}+q(x) y=0 .
$$

Let $\rho>0$ denote the minimum of the radii of convergence of the power series:

$$
p(x)=\sum_{n=0}^{\infty} p_{n} x^{n} \text { and } q(x)=\sum_{n=0}^{\infty} q_{n} x^{n}
$$

Let $r_{1}$ and $r_{2}$ be the real roots (we'll always be assuming our roots are real), of the indicial equation with $r_{1} \geq r_{2}$. Then
(1) For $x>0$, there exists a solution of our ODE of the form:

$$
y_{1}(x)=x^{r_{1}} \sum_{n=0}^{\infty} a_{n} x^{n}, a_{0} \neq 0
$$

corresponding to the larger root $r_{1}$.
(2) If $r_{1}-r_{2}$ is neither zero nor a positive integer, then there exists a second linearly independent solution for $x>0$ of the form:

$$
y_{2}(x)=x^{r_{2}} \sum_{n=0}^{\infty} b_{n} x^{n}, b_{0} \neq 0
$$

corresponding to the smaller root $r_{2}$.
The radii of convergence of $y_{1}$ and $y_{2}$ are at least $\rho$. We determine the coefficients by plugging our series into:

$$
x^{2} y^{\prime \prime}+x p(x) y^{\prime}+q(x) y=0
$$

Example - Use the method of Frobenius to solve the ODE:

$$
2 x^{2} y^{\prime \prime}+3 x y^{\prime}-\left(x^{2}+1\right) y=0
$$

around the regular singular point $x=0$.
Proof - Rewriting this we get:

$$
y^{\prime \prime}+\frac{\frac{3}{2}}{x} y^{\prime}+\frac{-\frac{1}{2}-\frac{1}{2} x^{2}}{x^{2}} y=0
$$

and so $p_{0}=\frac{3}{2}$ and $q_{0}=-\frac{1}{2}$.
This gives us the indicial equation:

$$
r(r-1)+\frac{3}{2} r-\frac{1}{2}=\left(r-\frac{1}{2}\right)(r+1)=0
$$

and so our two roots are $r_{1}=\frac{1}{2}$ and $r_{2}=-1$. So, our theorem guarantees two linearly independent Frobenius type solutions.

Frequently it's easier to work out our solutions without plugging in specific values of $r$ until the end. That's what we'll do here. Now, if we have a solution of the form:

$$
y(x)=\sum_{n=0}^{\infty} x^{n+r},
$$

then if we plug this form into our ODE we get the relation:

$$
\begin{gathered}
2 \sum_{n=0}^{\infty}(n+r)(n+r-1) c_{n} x^{n+r}+3 \sum_{n=0}^{\infty}(n+r) x^{n+r}-\sum_{n=0}^{\infty} c_{n} x^{n+r+2}- \\
\sum_{n=0}^{\infty} c_{n} x^{n+r}=0 .
\end{gathered}
$$

If we shift the third series over by 2 we get:

$$
\begin{gathered}
2 \sum_{n=0}^{\infty}(n+r)(n+r-1) c_{n} x^{n+r}+3 \sum_{n=0}^{\infty}(n+r) x^{n+r}-\sum_{n=2}^{\infty} c_{n-2} x^{n+r}- \\
\sum_{n=0}^{\infty} c_{n} x^{n+r}=0
\end{gathered}
$$

Now, if we look at the $x^{r}$ and $x^{r+1}$ coefficients we get the relations:

$$
\begin{gathered}
{[2 r(r-1)+3 r-1] c_{0}=2\left(r^{2}+\frac{1}{2} r-\frac{1}{2}\right) c_{0}=0} \\
{[2(r+1) r+3(r+1)-1] c_{1}=0}
\end{gathered}
$$

If we plug in our values for $r$ we see that the first of these is automatically satisfies for any $c_{0}$, as the multiplier of $c_{0}$ is just a constant multiplied by the indicial equation. On the other hand, if we plug in our values for $r$ we see that the second equation is only satisfies for $c_{1}=0$.

As for the other coefficients we get the relations:

$$
2(n+r)(n+r-1) c_{n}+3(n+r) c_{n}-c_{n-2}-c_{n}=0,
$$

which simplify to:

$$
c_{n}=\frac{c_{n-2}}{2(n+r)^{2}+(n+r)-1} \text { for } n \geq 2 .
$$

So, all the odd coefficients must be 0 given $c_{1}=0$. As for the even coefficients, for $r=\frac{1}{2}$ we get:

$$
a_{n}=\frac{a_{n-2}}{2 n^{2}+3 n},
$$

and for $r=-1$ we get:

$$
b_{n}=\frac{b_{n-2}}{2 n^{2}-3 n} .
$$

And, well, that's pretty much as good as we can do. If we write out our first few terms we get:

$$
y_{1}(x)=a_{0} x^{\frac{1}{2}}\left(1+\frac{x^{2}}{14}+\frac{x^{4}}{616}+\frac{x^{6}}{55,440}+\cdots\right)
$$

and

$$
y_{2}(x)=b_{0} x^{-1}\left(1+\frac{x^{2}}{2}+\frac{x^{4}}{40}+\frac{x^{6}}{2160}+\cdots\right)
$$

## 2. Method of Frobenius: The Exceptional Cases

Now, we have to take a look at what happens when $r_{1}-r_{2}$ is an integer. This could happen if $r_{1}=r_{2}$, or if $r_{1}=r_{2}+N$. In the latter case there might, or might not, be two Frobenius solutions. In the former case there's obviously only one Frobenius solution.

First, let's work an example where they differ by an integer but we get two distinct Frobenius solutions.

Example - Solve the ODE:

$$
x^{2} y^{\prime \prime}+\left(6 x+x^{2}\right) y^{\prime}+x y=0 .
$$

Solution - Here $p_{0}=6$ and $q_{0}=0$, and so our indicial equation is:

$$
r(r-1)+p_{0} r+q_{0}=r(r-1)+6 r=r(r+5),
$$

with roots $r=0$ and $r=-5$.
Now, if we substitute in from our Frobenius series we get:

$$
\begin{gathered}
\sum_{n=0}^{\infty}(n+r)(n+r-1) c_{n} x^{n+r}+6 \sum_{n=0}^{\infty}(n+r) c_{n} x^{n+r}+\sum_{n=0}^{\infty}(n+ \\
r) c_{n} x^{n+r+1}+\sum_{n=0}^{\infty} c_{n} x^{n+r+1}=0
\end{gathered}
$$

Combining these series we get:

$$
\sum_{n=0}^{\infty}\left[(n+r)^{2}+5(n+r)\right] c_{n} x^{n+r}+\sum_{n=1}^{\infty}(n+r) c_{n-1} x^{n+r}=0
$$

For $n=0$ as always we just get $c_{0}$ multiplied by the indicial equation, so for both of our possible values of $r c_{0}$ is arbitrary.

For $n \geq 1$ we get the relation:

$$
\left[(n+r)^{2}+5(n+r)\right] c_{n}+(n+r) c_{n-1} .
$$

If we begin with the smaller root we get:

$$
n(n-5) c_{n}+(n-5) c_{n-1}=0
$$

For $n \neq 5$ we have:

$$
c_{n}=-\frac{c_{n-1}}{n} .
$$

So, this gives us:

$$
\begin{gathered}
c_{1}=-c_{0}, c_{2}=-\frac{c_{1}}{2}=\frac{c_{0}}{2}, \\
c_{3}=-\frac{c_{2}}{3}=-\frac{c_{0}}{6}, c_{4}=-\frac{c_{3}}{4}=\frac{c_{0}}{24} .
\end{gathered}
$$

For $n=5$ we see that any choice of $c_{5}$ works, and so we get:

$$
\begin{gathered}
c_{5}=c_{5}, c_{6}=-\frac{c_{5}}{6}, \\
c_{7}=-\frac{c_{6}}{7}=\frac{c_{5}}{5 \times 6}, c_{8}=-\frac{c_{7}}{8}=-\frac{c_{5}}{6 \times 7 \times 8}, \text { etc } \ldots
\end{gathered}
$$

If we combine these we get the two solutions:

$$
\begin{gathered}
y_{1}(x)=x^{-5}\left(1-x+\frac{x^{2}}{2}-\frac{x^{3}}{6}+\frac{x^{4}}{24}\right) \\
\text { and } \\
y_{2}(x)=1+\sum_{n=1}^{\infty} \frac{(-1)^{n} x^{n}}{6 \times 7 \times \cdots \times(n+5)}=1+120 \sum_{n=1}^{\infty} \frac{(-1)^{n} x^{n}}{(n+5)!} .
\end{gathered}
$$

So, our solution is:

$$
y(x)=c_{1} y_{1}(x)+c_{5} y_{2}(x)
$$

Where we use the coefficient $c_{5}$ suggestively.
So, everything works great, right? Unfortunately not. Here's a situation where this method fails.

Example - Does the Frobenius method provide us two linearly independent solutions to the linear ODE:

$$
x^{2} y^{\prime \prime}-x y^{\prime}+\left(x^{2}-8\right) y=0 ?
$$

## Solution -

Here $p_{0}=-1, q_{0}=-8$, and so the indicial equation is:

$$
\phi(r)=r(r-1)-r-8=r^{2}-2 r-8=(r-4)(r+2)=0
$$

So, the roots of the inidial equation are -2 and 4 , two real numbers that differ by an integer.

If we substitute our Frobenius series into our ODE we get:

$$
\begin{gathered}
\sum_{n=0}^{\infty}(n+r)(n+r-1) c_{n} x^{n+r}-\sum_{n=0}^{\infty}(n+r) c_{n} x^{n+r}+\sum_{n=0}^{\infty} c_{n} x^{n+r+2}- \\
9 \sum_{n=0}^{\infty} c_{n} x^{n+r}=0
\end{gathered}
$$

Grouping these together we get:

$$
\sum_{n=0}^{\infty}\left[(n+r)^{2}-2(n+r)-8\right] c_{n} x^{n+r}+\sum_{n=0}^{\infty} c_{n-2} x^{n+r}=0
$$

The coefficient of $n=0$ is the indicial equation, and so the indicial equation can be anything. For $n=1$ we get:

$$
\left[(r+1)^{2}-2(r+1)-8\right] c_{1}=0
$$

Now, the polynomial is not 0 for $r=4$ or $r=-2$, and so $c_{1}=0$.
For $n \geq 2$ we get the relation:

$$
\left[(n+r)^{2}-2(n+r)-8\right] c_{n}+c_{n-2}=0
$$

For $r=-2$ we get:

$$
\begin{gathered}
n(n-6) c_{n}+c_{n-2}=0 \\
\text { and so } \\
c_{n}=-\frac{c_{n-2}}{n(n-6)}
\end{gathered}
$$

This works for $n \neq 6, n \geq 2$. This gives us:

$$
c_{0}=c_{0}, c_{2}=\frac{c_{0}}{8}, c_{4}=\frac{c_{0}}{64} .
$$

At $n=6$ we get:

$$
0 \cdot c_{6}+c_{4}=0 .
$$

If $c_{0} \neq 0$ then we're in trouble. So, we do not get two Frobenius solutions. We only get one, for $r=4$, which is:

$$
y_{1}(x)=x^{4}\left(1+6 \sum_{n=1}^{\infty} \frac{(-1)^{n} x^{2 n}}{2^{2 n} n!(n+3)!}\right)
$$

But, how do we find the other solution?
2.1. Reduction of Order. Suppose we have an ODE

$$
y^{\prime \prime}+P(x) y^{\prime}+Q(x) y=0
$$

and we've derived one solution $y_{1}(x)$. We want to find another solution $y_{2}(x)$.

First we note that finding $y_{2}(x)$ is equivalent to finding:

$$
v(x)=\frac{y_{2}(x)}{y_{1}(x)}
$$

Here we're assuming $y_{1}(x) \neq 0$ on our interval of interest. Now, if $y_{2}(x)=y_{1}(x) v(x)$ then:

$$
\begin{gathered}
y_{2}^{\prime}(x)=y_{1}(x) v^{\prime}(x)+y_{1}^{\prime}(x) v(x) \\
\text { and } \\
y_{2}^{\prime \prime}(x)=v(x) y_{1}^{\prime \prime}(x)+2 y_{1}^{\prime}(x) v^{\prime}(x)+y_{1}(x) v^{\prime \prime}(x)
\end{gathered}
$$

If we plug this into our ODE and group together we get:

$$
\begin{gathered}
{\left[v(x) y_{1}^{\prime \prime}(x)+2 v^{\prime}(x) y_{1}^{\prime}(x)+v^{\prime \prime}(x) y_{1}(x)\right]+P(x)\left[y_{1}(x) v^{\prime}(x)+\right.} \\
\left.y_{1}^{\prime}(x) v(x)\right]+Q(x) v(x) y_{1}(x)=0, \\
\Rightarrow v(x)\left(y_{1}^{\prime \prime}(x)+P(x) y_{1}^{\prime}(x)+Q(x) y_{1}(x)\right)+v^{\prime \prime}(x) y_{1}(x)+2 v^{\prime}(x) y_{1}^{\prime}(x)+ \\
P(x) v^{\prime}(x) y_{1}(x)=0 .
\end{gathered}
$$

Now, the first term is by definition 0 . This gives us the relation:

$$
v^{\prime \prime}(x) y_{1}(x)+\left(2 y_{1}^{\prime}(x)+P(x) y_{1}(x)\right) v^{\prime}(x)=0
$$

If we write $u(x)=v^{\prime}(x)$ then this becomes and equation that we know how to solve. Namely,

$$
u^{\prime}(x)+\left(2 \frac{y_{1}^{\prime}(x)}{y_{1}(x)}+P(x)\right) u(x)=0 .
$$

This is a first order ODE. So, if we introduct the integrating factor:

$$
\rho=e^{\int\left(2 \frac{y_{1}^{\prime}(x)}{y_{1}(x)}+P(x)\right) d x}=e^{2 \ln \left|y_{1}(x)\right|+\int P(x) d x}=\left(y_{1}(x)\right)^{2} e^{\int P(x) d x} .
$$

Integrating our ODE we get:

$$
u(x)\left(y_{1}(x)\right)^{2} e^{\int P(x) d x}=C
$$

If we note that $u(x)=v^{\prime}(x)$ and integrate for $v(x)$ we get:

$$
v(x)=C \int \frac{e^{-\int P(x) d x}}{\left(y_{1}(x)\right)^{2}} d x+K
$$

If we use $C=1$ and $K=0$ then we get:

$$
y_{2}(x)=y_{1}(x) \int \frac{e^{-\int P(x) d x}}{\left(y_{1}(x)\right)^{2}} d x .
$$

So, with one solution, we can find a second.
Example Find a second solution to the ODE:

$$
x^{2} y^{\prime \prime}-9 x y^{\prime}+25 y=0
$$

given that $x^{5}$ is one solution.
Solution

$$
\begin{gathered}
y_{2}(x)=x^{5} \int \frac{1}{\left(x^{5}\right)^{2}} e^{-\int-\frac{9}{x} d x} d x=x^{5} \int \frac{1}{x^{10}} e^{9 \ln |x|} d x=x^{5} \int \frac{x^{9}}{x^{10}} d x= \\
x^{5} \ln x .
\end{gathered}
$$

Now, how do we apply this method to our special Frobenius cases. Well, the textbook goes over a very long derivation that combines power series and the reduction of order method to figure out the form of the second solution. You can read it in the textbook if you'd like. The punchline is the following theorem.

Theorem - If you have the ODE:

$$
y^{\prime \prime}+\frac{p(x)}{x} y^{\prime}+\frac{q(x)}{x^{2}} y=0
$$

where $p(x)$ and $q(x)$ are analytic around $x=0$, and the root of the indicial equation, $r_{1}, r_{2}$ differ by an integer then one solution is:

$$
y_{1}(x)=x^{r_{1}} \sum_{n=0}^{\infty} c_{n} x^{n} .
$$

As for the other solution, if $r_{1}=r_{2}$, then:

$$
y_{2}(x)=y_{1}(x) \ln x+x^{r_{1}+1} \sum_{n=0}^{\infty} b_{n} x^{n} .
$$

On the other hand, if $r_{1} \neq r_{2}$, then:

$$
y_{2}(x)=C y_{1}(x) \ln x+x^{r_{2}} \sum_{n=0}^{\infty} b_{n} x^{n} .
$$

Here $C$ may or may not (we've seen examples of both) be 0 .
The radius of convergence of $y_{1}$, and $y_{2}$ is at least $\rho$, the minimum of the radii of convergence for $p(x)$ and $q(x)$.

Applying this theorem boils down to the same techniques we've seen before. Plug it in, find the recursion relations for the coefficients, and then see if you can get them into a closed form.


[^0]:    Date: Spring 2009.

