MATH 2280 - LECTURE 18

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1. Stability and the Phase Plane

A wide variety of natural phenomenon are modeled by a two-dimensional first-order system of the form

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

in which the independent variable t does not explicitly appear. Such systems, in which t does not appear, are often called *autonomous* systems. We will generally assume that the functions F and G are continuously differentiable. If this is the case then given any initial time t_0 and any initial point (x_0, y_0) of R, there is a unique solution $x = x(t)$, $y = y(t)$ that is defined on some open interval of "time" containing t_0. These equations describe a parametrized solution curve in the phase plane.

A *critical point* of the system is a point (x_*, y_*) such that:

$$F(x_*, y_*) = G(x_*, y_*) = 0.$$

If this is the case then a constant solution satisfies the differential equation. Namely, the solution that begins and then just stays at the critical point. This is called an *equilibrium solution.*

Example - Find the critical points of the given autonomous system:

$$\frac{dx}{dt} = x - 2y + 3$$
$$\frac{dy}{dt} = x - y + 2$$

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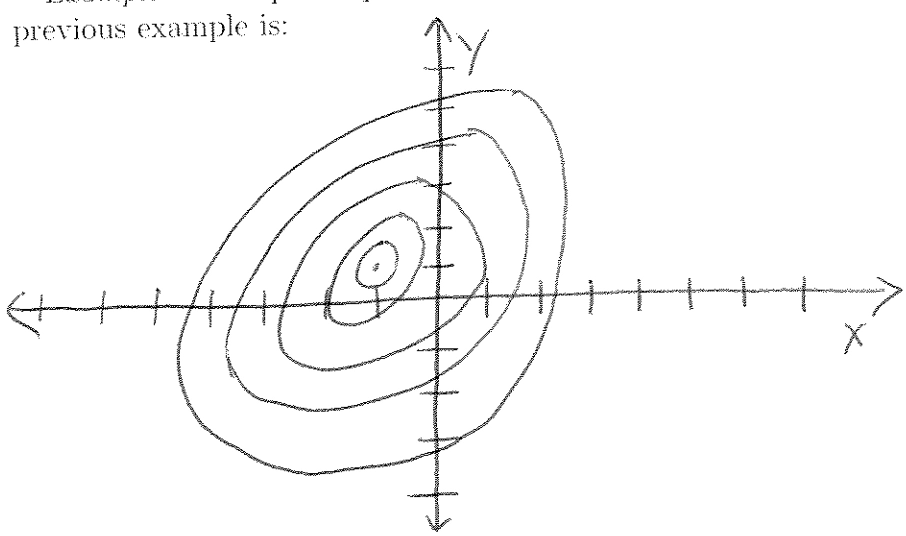
Solution - There will only be one critical point of this system, and that will be at $(-1, 1)$.

1.1. Phase Portraits.

If the initial point (x_0, y_0) is not a critical point, then the corresponding trajectory is a curve in the xy-plane. It turns out that such a curve will be a nondegenerate curve with no self intersections. A picture that shows an autonomous system's stable points, along with a collection of typical solution curves (trajectories) is called a *phase portrait*. We can also visualize this system by constructing a *slope field* in the xy-plane where each point has the slope:

$$\frac{dy}{dx} = \frac{G(x, y)}{F(x, y)}.$$

Example - The phase portrait for the differential equation in the previous example is:



1.2. Critical Point Behavior.

The behavior of an autonomous system near an isolated critical point is of particular interest. We'll take a look at some of the most common possibilities.

Consider the autonomous linear system:

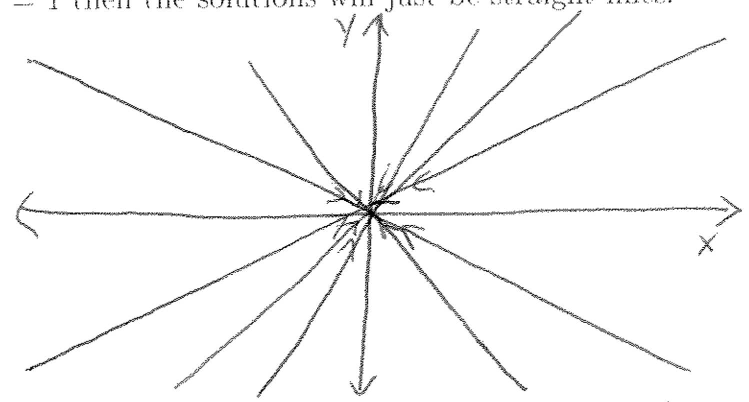
$$\frac{dx}{dt} = -x,$$

$$\frac{dy}{dt} = -ky.$$

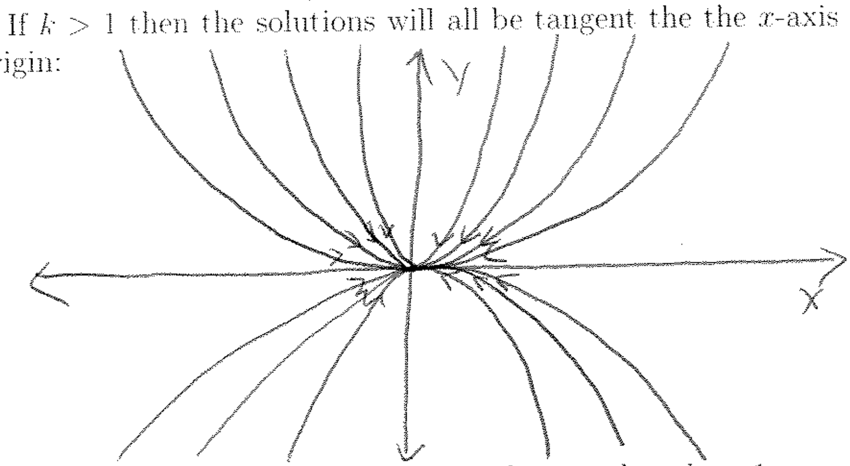
The only critical point for this system is at the origin, and the solution with initial point (x_0, y_0) is:

$$x(t) = x_0 e^{-t},\ y(t) = y_0 e^{-kt}.$$

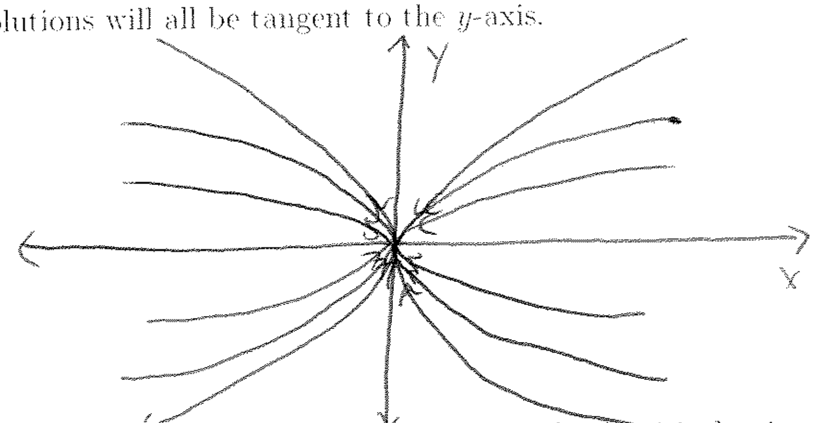
If $k = 1$ then the solutions will just be straight lines:



If $k > 1$ then the solutions will all be tangent the the x-axis at the origin:



If $0 < k < 1$ then it will look similar to when $k > 1$, except the solutions will all be tangent to the y-axis.



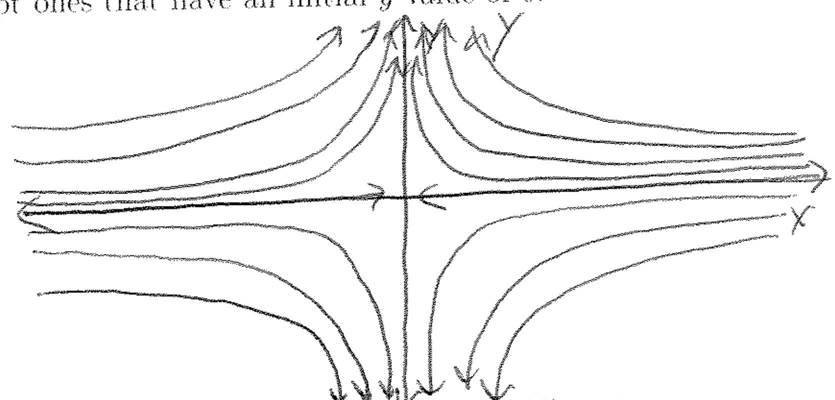
This type of critical point is called a *node*. A critical point is a node provided that:

- Either every trajectory approaches (x_*, y_*) as $t \to \infty$ or every trajectory recedes from (x_*, y_*) as $t \to \infty$.

- Every trajectory is asymptotically tangent to some straight line through the critical point, either as $t \to \infty$ for a sink, or as $t \to -\infty$ for a source. These terms are defined in the next paragraph.

A node is said to be *proper* if no two different pairs of "opposite" trajectories are tangent to the same line through the critical point. A node is *improper* if it is not proper. A node is called a *sink* if all trajectories approach it, and a *source* if all trajectories recede from it.

In the above example if $k < 0$ then every trajectory recedes from it except ones that have an initial y value of 0.



This type of critical point is called a *saddle point.*

1.3. **Stability.** A critical point of an autonomous system is said to be *stable* provided that if the initial point is sufficiently close to the critical point, it stays close to the critical point. In mathematical terminology, we say that the point $\mathbf{x}_*$ is stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$|\mathbf{x}_0 - \mathbf{x}_*| < \delta \text{ implies that } |\mathbf{x}(t) - \mathbf{x}_*| < \epsilon$$

for all $t > 0$. A critical point is *unstable* if it is not stable. The earlier example for $k > 0$ had the origin as a stable critical point, while for $k < 0$ the origin was an unstable critical point.

Example - The system we examined in the first example is a a stable system, but we note that it's not a system in which as $t \to \infty$ trajectories close to the stable point approach the stable point.

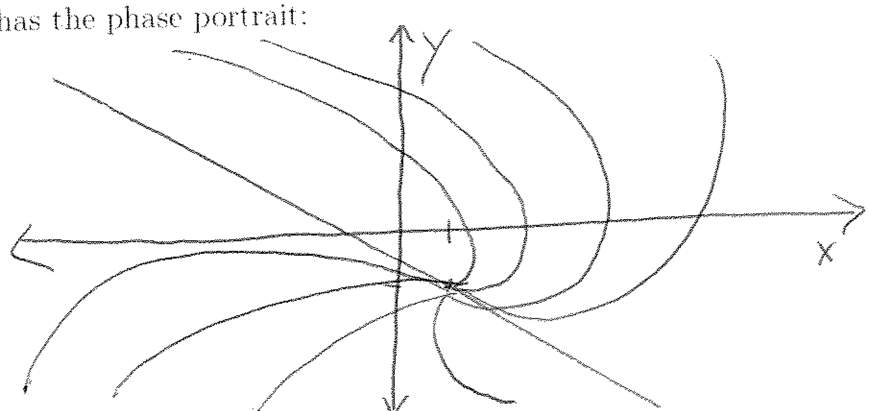
1.4. **Asymptotic Stability.** The critical point (x_*, y_*) is called *asymptotically stable* if it is stable and every trajectory that begins sufficiently close to (x_*, y_*) approaches (x_*, y_*) as $t \to \infty$. In mathematical terms we say that there exists a $\delta > 0$ such that:

$$|\mathbf{x}_0 - \mathbf{x}_*| < \delta \text{ implies that } \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_*.$$

Example - The system:

$$\frac{dx}{dt} = 2x - 2y - 4$$
$$\frac{dy}{dt} = x + 4y + 3.$$

has the phase portrait:

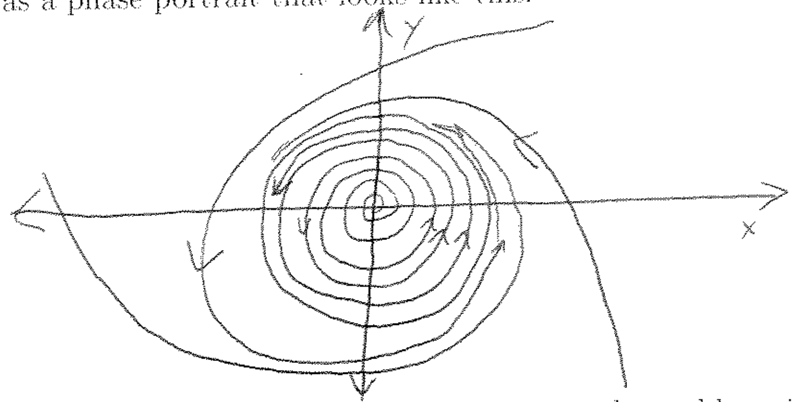


The critical point $(1, -1)$ is a spiral point, which is asymptotically stable.

Finally, we note that the only possibilities are not descent into a stable point, a closed trajectory, or unbounded increasing. It's also possible that the solution approaches a closed trajectory. For example, the system:

$$\frac{dx}{dt} = -ky + x(1 - x^2 - y^2).$$
$$\frac{dy}{dt} = kx + y(1 - x^2 - y^2).$$

Has a phase portrait that looks like this:



where we see that any trajectory except at the stable point (0, 0) approaches a trajectory around the unit circle.

In general the four possibilities for a critical point are:

(1) The trajectory approaches a critical point as $t \to \infty$.
(2) The trajectory is unbounded with increasing t.
(3) The trajectory is a periodic solution with a closed trajectory.
(4) The trajectory spirals towards a closed trajectory as $t \to \infty$.