MATH 2280 - LECTURE 17

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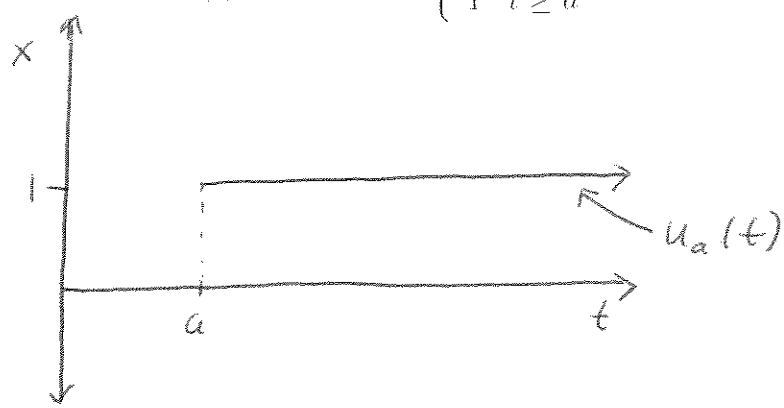
1. Periodic and Piecewise Continuous Input Functions

We start this discussion by examining a very simple function, the unit step function, which is defined by:

$$u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



Now, if we calculate the Laplace transform of this equation we get:

$$\mathcal{L}(u(t-a)) = \int_0^\infty e^{-st}u(t-a)dt = \int_a^\infty e^{-st}dt$$
$$= \int_0^\infty e^{-s(t+a)}dt = e^{-as}\int_0^\infty e^{-st}dt = \frac{e^{-as}}{s}$$

So, we see that multiplication of the transform of $u(t)$ by e^{-as} corresponds to the translation $t \to t-a$ in the original independent variable.

Now, before going any further I want to point out that the Laplace transform of a function $f(t)$, and the Laplace transform of the function $u(t)f(t)$ are in fact *the same thing*, in that when dealing with the Laplace transform we're completely uninterested in what happens for negative values of the function.

Now, you may say that this apathy sounds a bit ridiculous. After all, there's a big difference between this:

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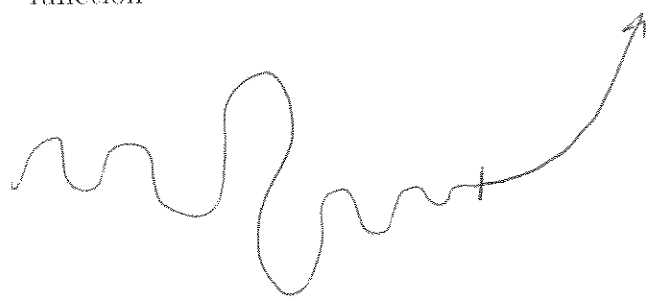
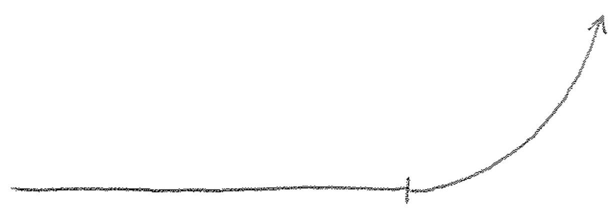
function



and this:



and in general, you'd be right. When we're using Laplace transforms we're usually either assuming that our solution begins at some set time, or we calculate a solution that works for non-negative values, and then extend this solution to the entire real line and confirm that it works. In many physical applications this is a very reasonable assumption, because most things "begin", and few things have been going on "forever".

Getting back to step functions, we've seen that, in the case of the step function, multiplying the Laplace transform by e^{-as} is the same as translating the original function over a distance a. This relation holds in general:

Theorem - If $\mathcal{L}(f(t))$ exists for $s > c$, then

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s)$$

and conversely

$$\mathcal{L}^{-1}(e^{-as}F(s)) = u(t-a)f(t-a)$$

for $s > c + a$.

The proof of this I'll leave as an exercise. It follows directly from the definition of the Laplace transform.

Example Calculate the inverse Laplace transform of:

$$F(s) = \frac{e^{-s} - e^{-3s}}{s^2}$$

Solution

$$\mathcal{L}^{-1}\left(\frac{e^{-s} - e^{-3s}}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2}\right)$$
$$= u(t-1)(t-1) - u(t-3)(t-3).$$

Example Calculate the Laplace transform of the function:

$$f(t) = \begin{cases} 2 & 0 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

Solution
We see that this is the function $f(t) = 2 - 2u(t-3)$, which will have the Laplace transform:

$$\mathcal{L}(f(t)) = \frac{2}{s} - \frac{2e^{-3s}}{s}.$$

Example Calculate the Laplace transform of the function:

$$f(t) = \begin{cases} \sin t & 0 \leq t \leq 3\pi \\ 0 & t > 3\pi \end{cases}$$

Solution
This is the function $f(t) = \sin t - u(t-3\pi)\sin t$, which is the same thing as $f(t) = \sin t + u(t-3\pi)\sin(t-3\pi)$, which will have the Laplace transform:

$$\mathcal{L}(f(t)) = \frac{1}{s^2+1} + \frac{e^{-3\pi s}}{s^2+1} = \frac{1+e^{-3\pi s}}{s^2+1}.$$

1.1. **Transforms of Periodic Functions.** We say a function defined for $t \geq 0$ is periodic if there is a number $p > 0$ such that

$$f(t+p) = f(t)$$

for all $t \geq 0$. The least positive value of p (if any) for which the equation holds is called *the* period of the function f. If a function is periodic, it's actually relatively easy to calculate its Laplace transform, and doesn't require the computation of an indefinite integral.

Theorem - Let $f(t)$ be periodic with period p and piecewise continuous for $t \geq 0$. Then the transform $F(s) = \mathcal{L}(f(t))$ exists for $s > 0$ and is given by

$$F(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt.$$

The proof of this theorem is kind of fun, so let's go over it.

Proof - The definition of the Laplace transform gives

$$F(s) = \int_0^\infty e^{-st} f(t) dt = \sum_{n=0}^\infty \int_{np}^{(n+1)p} e^{-st} f(t) dt.$$

Now, we note that if we use our periodicity property and the substitution $\tau = t + np$ we get:

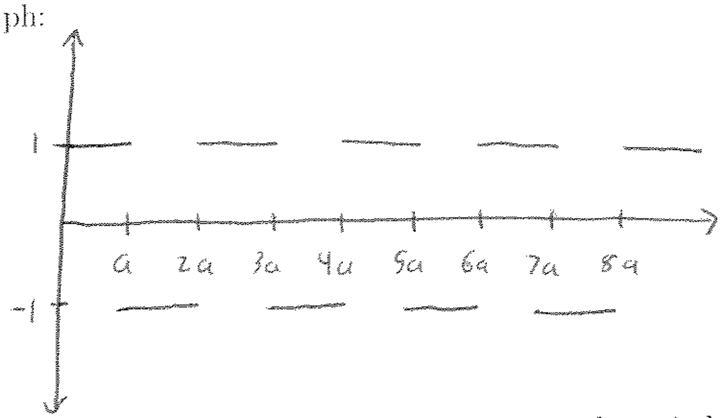
$$\int_{np}^{(n+1)p} e^{-st} f(t) dt = \int_0^p e^{-s(\tau + np)} f(\tau + np) d\tau = e^{-nps} \int_0^p e^{-s\tau} f(\tau) d\tau.$$

So, using this relation, we see that our Laplace transform is:

$$F(s) = \sum_{n=0}^\infty \left(e^{-nps} \int_0^p e^{-s\tau} f(\tau) d\tau \right) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-s\tau} f(\tau) d\tau.$$

Example Calculate the Laplace transform of the square-wave function $f(t) = (-1)^{[t/a]}$ of period $p = 2a$, where $[x]$ denotes the greatest integer not exceding x.

Graph:



Solution As mentioned, this is a function with period $2a$, and so applying our formula we get:

$$F(s) = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t) dt$$

$$F(s) = \frac{1}{1-e^{-2as}} \left(\int_0^a e^{-st} dt + \int_a^{2a} (-1) e^{-st} dt \right)$$

$$= \frac{1-2e^{-as}+e^{-2as}}{s(1-e^{-2as})} = \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})}$$

$$= \frac{1}{s} \tanh \frac{as}{2}.$$

Example - Apply this theorem to verify that $\mathcal{L}(\cos kt) = s/(s^2+k^2)$.

Solution This function is periodic with period $2\pi/k$, and so the Laplace transform will be the integral:

$$\mathcal{L}(\cos kt) = \frac{1}{1-e^{2\pi s/k}} \int_0^{2\pi/k} e^{-st} \cos(kt) dt.$$

Now, if we use the relation:

$$\int e^{-st} \cos(kt) dt = \frac{e^{-st}}{k^2+s^2}(k \sin kt - s \cos kt),$$

we get the solution:

$$\mathcal{L}(\cos kt) = \frac{1}{1-e^{2\pi s/k}} e^{-st} \left(\frac{k \sin kt - s \cos kt}{k^2+s^2} \right) \Big|_0^{2\pi/k} =$$

$$\frac{s}{k^2+s^2} \frac{1-e^{-2\pi s/k}}{1-e^{-s\pi s/k}} = \frac{s}{k^2+s^2}.$$

which is what we wanted. Score!

2. Impulse and Delta Functions

Consider a force $f(t)$ that acts only during a very short time interval $a \leq t \leq b$, with $f(t) = 0$ outside this interval. A bat striking a ball or a bolt of lightning striking a tower, for example. Typically, the effect of this force depends only on the integral:

$$p = \int_a^b f(t)dt.$$

This number is called the *impulse* of the force $f(t)$ over the interval $[a, b]$.

An example of this is that the change in momentum of a particle is determined by the impulse of the force acting upon it.

This is nice because frequently we don't know exactly what the force $f(t)$ is, but we can figure out what the integral above, the impulse, is, and it turns out that that's really all we need to know.

Now, if we have a given impulse p, we may as well model it with the simplest function we can over the interval, namely, a constant function. So, if we have an impulse $p = 1$, we can get this same impulse using the function:

$$d_{a,\epsilon}(t) = \begin{cases} \frac{1}{\epsilon} & a \leq t < a + \epsilon \\ 0 & otherwise \end{cases}$$

where ϵ models the amount of time the impulse acts. We see, if $a > 0$, that

$$\int_0^\infty d_{a,\epsilon}(t)dt = 1.$$

Now, the time interval ϵ over which the impulse acts are frequently *very* small, and it's difficult to get a good measure of what this time is. So, we can try to model an *instantaneous impulse* that occurs preciesely at the time $t = a$. We call this instantaneous impulse the *Dirac delta funciton*, and we represent it as:

$$\delta_a(t) = \lim_{\epsilon \to 0} d_{a,\epsilon}(t).$$

Now, this delta function isn't a "function" in the strictest sense. It's 0 everywhere except at the point a, and at a it's ∞. Now, ∞ isn't well defined, and a function that is 0 everywhere except at a point should integrate to 0 over any finite interval. Instead, this "function" is actually a generalized function called a distribution, and is only defined in terms of how it operates on integrals.

2.1. **Delta Functions as Operators.** The mean value theorem for integral states that:

$$\int_a^{a+\epsilon} g(t)dt = \epsilon g(\overline{t})$$

where $\overline{t}$ is a point in $[a, a+\epsilon]$. It follows that:

$$\lim_{\epsilon \to 0} \int_0^\infty g(t) d_{a,\epsilon}(t) dt = \lim_{\epsilon \to 0} \int_a^{a+\epsilon} g(t) \cdot \frac{1}{\epsilon} dt = \lim_{\epsilon \to 0} g(\overline{t}) = g(a).$$

by continuity of t at $t = a$. Now, we take this as the *definition* of the Delta function. It's an operator such that:

$$\int_0^\infty f(t)\delta_a(t)dt = f(a).$$

Now, we note that if $f(t) = e^{-st}$ we get:

$$\int_0^\infty e^{-st}\delta_a(t)dt = e^{-as}.$$

We *define* the Laplace transform of the delta function to be:

$$\mathcal{L}(\delta_a(t)) = e^{-as} \ (a \geq 0).$$

Now, if $a = 0$ we get:

$$\mathcal{L}(\delta(t)) = 1.$$

We note that as $s \to \infty$ this Laplace transform does *not* go to 0, a further implication that the Delta function is not a standard type of function.

2.2. **Delta Function Inputs.** Finally, suppose that we are given a mechanical system whose response $x(t)$ to the external force $f(t)$ is determined by the differential equation:

$$Ax'' + Bx' + Cx = f(t).$$

Now, if we want to investigate the response of this system to a unit impulse at the time $t = a$, it seems reasonable to express this response as the solution to the differential equation:

$$Ax'' + Bx' + Cx = \delta_a(t).$$

But, again, $\delta_a(t)$ isn't really a function, and so what would we mean by a solution to the above equation? We call $x(t)$ a solution to the above differential equation provided that:

$$x(t) = \lim_{\epsilon \to 0} x_\epsilon(t),$$

where $x_\epsilon(t)$ is a solution of the differential equation:

$$Ax'' + Bx' + Cx = d_{a,\epsilon}(t).$$

Now, it turns out that the way to get this solution $x(t)$ is to just take the Laplace transform of both sides, figure out $X(s)$, and then figure out its inverse Laplace transform. This is how we solve these type of differential equations, and it's the first real instance we've seen where Laplace transform methods are necessary.

Example Solve the initial value problem:

$$x'' + 4x = \delta(t) + \delta(t - \pi);$$
$$x(0) = x'(0) = 0.$$

Solution - If we take the Laplace transform of both sides we get:

$$s^2X(s) + 4X(s) = 1 + e^{-\pi s}$$

and solving this for $X(s)$ we get:

$$X(s) = \frac{1 + e^{-\pi s}}{s^2 + 4}$$

from which we can calculate the inverse Laplace transform using our tables of Laplace transforms:

$$x(t) = \frac{1}{2}\sin(2t) + \frac{1}{2}u(t - \pi)\sin(2(t - \pi)) = \frac{1}{2}\sin(2t)(1 + u(t - \pi)).$$

Example - Solve the initial value problem:

$$x'' + 2x' + x = t + \delta(t);$$
$$x(0) = 0,\ x'(0) = 1.$$

Solution - Taking the Laplace transform of both sides we get:

$$s^2X(s) - 1 + 2sX(s) + X(s) = \frac{1}{s^2} + 1$$

which, if we solve for $X(s)$, we get:

$$X(s) = \frac{1}{s^2(s+1)^2} + \frac{2}{(s+1)^2}$$

Now, if we solve this using a partial fraction decomposition we get:

$$X(s) = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{3}{(s+1)^2}$$

which has the inverse Laplace transform:

$$x(t) = -2 + t + 2e^{-t} + 3te^{-t}.$$

2.3. **Systems Analysis and Duhamel's Principle.** Consider a physical system in which the output $x(t)$ to the input function $f(t)$ is described by the differential equation:

$$ax'' + bx' + cx = f(t)$$

where the constant coefficients a, b and c are determined by the physical parameters of the system and are independent of $f(t)$. We assume for simplicity that the system is initially passive, and so $x(0) = x'(0) = 0$. Then the transform of our differential equation is:

$$as^2X(s) + bsX(s) + cX(s) = F(s)$$

and so our Laplace transform $X(s)$ is then given by:

$$X(s) = \frac{F(s)}{as^2 + bs + c} = W(s)F(s).$$

Here the function

$$W(s) = \frac{1}{as^2 + bs + c}$$

is called the *transfer function* of the system. The inverse Laplace transform of the transform function, $w(t)$, is called the *weight function* of the system. Using our earlier result about convolutions and the above formula for $X(s)$, we see that the solution to our system is:

$$x(t) = \int_0^t w(\tau)f(t-\tau)dt.$$

This is called *Duhamel's principle* for the system, and the important thing about it is that the weight function $w(t)$ is determined completely by the parameters of the system, and has nothing to do with the imput function $f(t)$. So, if we know the weight function, we can calculate the solution for *any* input by "just" calculting an integral. Now, integrals aren't easy, but they're easier that solving differential equations. Now, it's interesting (actually, it's *very* interesting, for reasons we won't explore in this class) that our weight function is actually the response of our system to a delta function input.

Example - Apply Duhamel's principle to write an integral formula for the solution of the initial value problem:

$$x'' + 6x' + 9x = f(t);$$
$$x(0) = x'(0) = 0.$$

Solution - The transfer function of this system is:

$$X(s) = \frac{1}{s^2 + 6s + 9} = \frac{1}{(s+3)^2}.$$

The inverse Laplace transform of this transfer function, the weight function, will be:

$$w(t) = te^{-3t}.$$

So, the response $x(t)$ will be given by the integral equation:

$$x(t) = \int_0^t \tau e^{-3\tau} f(t-\tau)d\tau.$$