Math 2280 - Lecture 16

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Spring 2009

1 Translation and Partial Fractions

Today we'll go over some methods for finding inverse Laplace transforms for forms you'll see frequently as Laplace transforms. The major example of this type of forms is a rational function of the form:

$$R(s) = \frac{P(S)}{Q(s)}$$

where the degree of $P(s)$ is less than that of $Q(s)$. The idea here is that we want to use a partial fraction decomposition. In other words, we want to factor $Q(s)$ into its linear and quadratic parts (which can always in theory be done for polynomials with real coefficients) and then based upon these parts write our quotient as a sun according to two rules.

Rule 1 - Linear Factor Partial Fractions

The portion of the partial fraction decomposition of $R(s)$ corresponding to the linear factor $(s-a)$ (where we mean a linear factor of $Q(s)$) of multiplicyt n is a sum of n partial fractions, having the form:

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$

where $A_1, A_2, \ldots, A_n$ are constants.

Rule 2 Quadratic Factor Partial Fractions

The portion of the partial fraction decomposition corresponding to the irreducible quadratic factor $(s-a)^2+b^2$ of multiplicity n is a sum of n partial fractions, having the form:

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{[(s-a)^2+b^2]^2} + \cdots + \frac{A_n s + B_n}{[(s-a)^2+b^2]^n},$$

where $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$ are constants.

After we find the partial fraction decomposition we then use the partial fraction decomposition to find the inverse Laplace transform.

The formulas and relations that we're going to find most useful are:

Theorem - If $F(s) = \mathcal{L}(f(t))$ exists for $s > c$, then $\mathcal{L}(e^{at}f(t))$ exists for $s > a + c$, and

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

or equivalently

$$\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t)$$

Proving this is very simple indeed:

$$F(s-a) = \int_0^\infty e^{-(s-a)t}f(t)dt = \int_0^\infty e^{-st}[e^{at}f(t)]dt = \mathcal{L}(e^{at}f(t)).$$

This fact, combined with our relations:

$$\mathcal{L}(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}},\ s > a;$$

$$\mathcal{L}(e^{at}\cos kt) = \frac{s-a}{(s-a)^2+k^2},\ s > a;$$

$$\mathcal{L}(e^{at}\sin kt) = \frac{k}{(s-a)^2+k^2}, \, s > a;$$

$$\mathcal{L}\left(\frac{e^{at}}{2k}t\sin kt\right) = \frac{s-a}{((s-a)^2+k^2)^2};$$

$$\mathcal{L}\left(\frac{e^{at}}{2k^3}(sinkt - kt\cos kt)\right) = \frac{1}{((s-a)^2+k^2)^2};$$

will allow us to figure out the inverse Laplace transform given almost any partial fraction decomposition. I say almost any because you might have a quadratic term to a higher than second power. In this case I'd say, first, that in this class you'll never see a quadratic power greater than 2, and second we'll go over how you can calculate those Laplace transforms using convolutions later on this lecture. So, we'll see that, in theory, the inverse Laplace transform for any partial fraction decomposition can be calculated. It just might take a while. We note that a repeated quadratic factor usually corresponds to a situation where we have resonance in our system.

Example Solve the initial value problem:

$$y'' + 4y' + 4y = t^2; \, y(0) = y'(0) = 0.$$

Now, we could solve this using techniques we already know, namely the method of undetermined coefficients, but let's figure this out using Laplace transforms. If we take the Laplace transform of both sides we get:

$$s^2Y(s) + 4sY(s) + +4Y(S) = \frac{2}{s^3}.$$

If we then solve this for $Y(s)$ we get:

$$Y(s) = \frac{2}{s^3(s^2+4s+4)} = \frac{2}{s^3(s+2)^2}.$$

If we calculate the partial fraction decomposition of this we know it will be of the form:

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{(s+2)} + \frac{E}{(s+2)^2}.$$

If we then solve this for the unknowns (equate the coefficients, do some linear algebra...) we get:

$$Y(s) = \frac{3/8}{s} - \frac{1/2}{s^2} + \frac{1/2}{s^3} - \frac{3/8}{s+2} - \frac{1/4}{(s+2)^2}.$$

This will have the inverse Laplace transform:

$$y(t) = \frac{3}{8} - \frac{1}{2}t + \frac{1}{4}t^2 - \frac{3}{8}e^{-2t} - \frac{1}{4}te^{-2t}$$

which is our desired solution. Sweet!

Example - Apply the translation to find the Laplace transform of:

$$f(t) = e^{-2t}\sin 3\pi t$$

We just look at our table to get:

$$\mathcal{L}(f(t)) = \frac{3\pi}{(s+2)^2 + 9\pi^2}.$$

Example - Calculate the inverse Laplace transform of the function:

$$F(s) = \frac{s-1}{(s+1)^2}$$

Solution

It becomes more straightforward if we rewrite it as:

$$\frac{s-1}{(s+1)^2} = \frac{s+1}{(s+1)^2} - \frac{2}{(s+1)^2} = \frac{1}{s+1} - \frac{2}{(s+1)^2}.$$

Then, we can just read the inverse transforms from our list of standard inverse transforms:

$$f(t) = e^{-t} - 2te^{-t} = (1-2t)e^{-t}.$$

Example - Calculate the inverse Laplace transform of the function:

$$F(s) = \frac{1}{(s^2+s-6)^2}$$

Solution

If we factor the denominator we get:

$$F(s) = \frac{1}{(s+3)^2(s-2)^2}$$

which will have the partial fraction decomposition:

$$\frac{1}{(s+3)^2(s-2)^2} = \frac{A_1}{s+3} + \frac{A_2}{(s+3)^2} + \frac{A_3}{s-2} + \frac{A_4}{(s-2)^2}.$$

This would then imply:

$$1 = A_1(s+3)(s-2)^2 + A_2(s-2)^2 + A_3(s+3)^2(s-2) + A_4(s+3)^2.$$

If we plug in $s = 2$ we get $A_4 = \frac{1}{25}$. If we plug in $s = -3$ we get $A_2 = \frac{1}{25}$. After some algebra we can get $A_1 = \frac{2}{125}$ and $A_3 = -\frac{2}{25}$. Pluggint these values in we get:

$$\frac{1}{(s+3)^2(s-2)^2} = \frac{\frac{2}{125}}{s+3} + \frac{\frac{1}{25}}{(s+3)^2} - \frac{\frac{2}{125}}{(s-2)} + \frac{\frac{1}{25}}{(s-2)^2}.$$

From here we can again read off the inverse Laplace transform from our tables in a straightforward way:

$$f(t) = \frac{2}{125}e^{-3t} + \frac{1}{25}te^{-3t} - \frac{2}{125}e^{2t} + \frac{1}{25}te^{2t}.$$

2 Derivatives, Integrals, and Products of Transforms

Now, we know that the Laplace transform is linear, but what do we know about how it handles products? Well, it would be "nice" if:

$$\mathcal{L}(f(t) \cdot g(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t))$$

but this is *NOT TRUE*. In fact, we can see it's obvious not true given that $\mathcal{L}(t) = 1/s \neq 1$. In other words, if this were true then we'd have:

$$\tfrac{1}{s} = \mathcal{L}(1) = \mathcal{L}(1 \cdot 1) = \mathcal{L}(1) \cdot \mathcal{L}(1) = \tfrac{1}{s^2}.$$

This is ridiculous, so we must state that in general:

$$\mathcal{L}(f(t) \cdot g(t)) \neq \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t)).$$

So, we return to our question. Namely, what, if any, relation is there between products and Laplace transforms?

To answer this question we first need to define an operator called *convolution*. The convolution of two functions $f(t)$ and $g(t)$ is denoted as $f(t) * g(t)$, and is defined by:

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

We note that this operator it's a straightforward if tedious task to verify that the operator is commutative and associative.

Example - Calculate the convolution:

$$t^2 * \cos t.$$

Solution

The convolution is:

$$\int_0^t \tau^2 \cos(t-\tau)d\tau$$

Using the relation:

$$\cos(t-\tau) = \cos t \cos\tau + \sin t \sin\tau$$

we get that this is equal to:

$$\cos t \int_0^t \tau^2 \cos\tau d\tau + \sin t \int_0^t \tau^2 \sin\tau$$

If we use a table of integrals or integrals.com we get:

$$\int \tau^2 \cos\tau d\tau = \tau^2 \sin\tau + 2\tau\cos\tau + 2\sin\tau$$

and

$$\int \tau^2 \sin\tau d\tau = -\tau^2 \cos\tau + 2\tau\sin\tau - 2\cos\tau.$$

And so our integrals are:

$$\cos t \int_0^t \tau^2 \cos \tau d\tau = t^2 \sin t \cos t + 2t \cos^2 t + 2 \sin t \cos t$$

$$\sin t \int_0^t \tau^2 \sin \tau d\tau = -t^2 \sin t \cos t + 2t \sin^2 t - 2 \sin t \cos t + 2 \sin t.$$

Therefore, our convolution is the sum of the above:

$$t^2 + 2 \sin t.$$

Now, why are these convolutions important? Well, as you might expect, they relate to products of Laplace transforms in a big way. This relation is given by our next theorem.

Theorem - Suppose that $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$ and that $|f(t)|$ and $|g(t)|$ are bounded by Me^{ct} as $t \rightarrow \infty$. Then the Laplace transform of $f * g$ exists for $s > c$, and

$$\mathcal{L}(f(t) * g(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t)).$$

Example - Calculate the inverse Laplace transform of:

$$F(s) = \frac{1}{s(s^2+1)}.$$

Solution

We have that:

$$F(s) = \frac{1}{s(s^2+1)} = \left(\frac{1}{s}\right)\left(\frac{1}{s^2+1}\right) = \mathcal{L}(1) \cdot \mathcal{L}(\sin t).$$

So, the inverse Laplace transform of this is given by:

$$f(t) = 1 * \sin t = \int_0^t \sin \tau d\tau = 1 - \cos t.$$

Now, using this theorem we can derive the following relations:

$$\mathcal{L}(-t \cdot f(t)) = F'(s)$$

and in greater generality

$$\mathcal{L}(t^n \cdot f(t)) = (-1)^n F^{(n)}(s).$$

Also, conversely

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma) d\sigma.$$

Example - Calculate the Laplace transform:

$$\mathcal{L}(t \sin(3t)).$$

Solution

Using our relation we get that this is equal to:

$$-F'(s), \text{ where } F(s) = \mathcal{L}(\sin(3t)) = \frac{3}{s^2+9}.$$

Taking the derivative we get:

$$\mathcal{L}(t \sin(3t)) = \frac{6s}{(s^2+9)^2}.$$

Example - Calculate the Laplace transform:

$$\mathcal{L}\left(\frac{e^{3t}-1}{t}\right).$$

Solution

We know the Laplace transform:

$$\mathcal{L}(e^{3t}-1) = \frac{1}{s-3} - \frac{1}{s}.$$

So,

$$\mathcal{L}\left(\frac{e^{3t}-1}{t}\right) = \int_s^\infty \left(\frac{1}{\sigma-3} - \frac{1}{\sigma}\right) d\sigma.$$

Calculating this integral we get:

$$\int_s^\infty \left(\frac{1}{\sigma-3} - \frac{1}{\sigma}\right) d\sigma = \ln(\sigma-3) - \ln\sigma|_s^\infty$$
$$= \ln\left(\frac{\sigma-3}{\sigma}\right)|_s^\infty == -\ln\left(\frac{s-3}{s}\right) = \ln\left(\frac{s}{s-3}\right) \text{ for } s > 3.$$