Math 2280 - Lecture 15

Dylan Zwick

Spring 2009

1 Laplace Transforms and Inverse Problems

We will now move into the study of Laplace transforms and their relation with differential equations. A Laplace transformation is a map from functions to functions, kind of like differentiation. For example, the differential operator would map the function $f(t) = t^2$ to:

$$D_t(f(t)) = 2t = f'(t).$$

So, we have an operator, the differential operator, that takes a function $f(t)$ as its input, and outputs another function. Now, this operator is not well defined for all functions. In fact, it's only defined on a particular class of functions called, not surprisingly, the differentiable functions.

The Laplace transform is a similar type of operation, in that it takes a function as its input and outputs another function. There's a certain set of functions that are in the domain (allowable inputs) of the Laplace operator, and so not every function has a well-defined Laplace transform.

For our earlier example, we'd find that the Laplace transform of t^2 is:

$$\mathcal{L}(t^2) = \frac{2}{s^3}.$$

To see why this is so, we need to see how the Laplace transform is defined.

1.1 Definitions and Properties

The Laplace transform is defined as:

Definition - Given a function $f(t)$ defined for all $t \geq 0$, the *Laplace transform* of f is the function F defined as:

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

for all values of s for which the improper integral converges.

We recall that for an inproper integral what we mean by a limit of integration being infinity is that it's defined in terms of the limit:

$$\int_0^\infty g(t)dt = \lim_{b \to \infty} \int_0^b g(t)dt$$

and this only makes sense if the limit is well defined.

Let's do some examples, beginning with the example stated above.

Example - The Laplace transform of t^2 is:

$$\mathcal{L}(t^2) = \int_0^\infty t^2 e^{-st} dt = -\frac{t^2}{s} e^{-st} - \frac{2t}{s^2} e^{-st} - \frac{2}{s^3} e^{-st} |_0^\infty$$

where we get the above result using integration by parts twice. Now, if $s \leq 0$ then the above integral diverges, while if $s > 0$ we have convergence to our values:

$$\mathcal{L}(t^2) = \frac{2}{s^3}$$

and so our Laplace transform is only defined for $s > 0$. Something like this will be true in almost every case that we deal with, and so we'll almost always have to specify the domain upon which our Laplace transform is well defined.

Now, just based upon the definition and the linearity of integration we can deduce that the Laplace transform, like differentiation, is a linear operator:

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

where $f(t)$ and $g(t)$ are functions of t and a, b are constants.

1.2 Some Common Laplace Transforms

For starters, let's calculate the Laplace transform of the function $f(t) = 1$:

$$\mathcal{L}(1) = \int_0^\infty e^{-st}dt = \frac{e^{-st}}{s}|_0^\infty = \frac{1}{s}$$

where it is only well defined for $s > 0$.

Well, just using the two Laplace transforms we've calculated so far, along with our linearity property, we can figure out some other Laplace transforms without resorting back to our definition:

$$\mathcal{L}(3t^2 + 5) = \frac{6}{s^3} + \frac{5}{s}$$
$$s > 0$$

This is something we're going to want to do frequently. Again, in analogy with differentiation, calculating a Laplace transform using the formal definition can be a major pain, and we're going to want to figure out the Laplace transforms of some common functions, and some basic rules for how we deal with sums and products of these functions, and then use these rules as frequently as we can to calculate our Laplace transforms. It saves a lot of time.

Let's talk about how to take Laplace transforms of functions of the form t^a, where a is a real number and $a > -1$. Well, to do this we're going to first want to define a very interesting function called the *gamma function*, which is defined for $x > 0$ by:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Now, it's the matter of some simple integration to check that:

$$\Gamma(1) = 1$$

and using integration by parts we get the relation:

$$\Gamma(x+1) = x\Gamma(x)$$

Now, this is interesting. If we say that x is an integer n we get that:

$$\Gamma(n+1) = n!$$

I'd like to put an exclamation mark at the end of that sentence, but there's already one there. In fact, this is how we technically want to *define* the factorial relation, and it gives an explanation as to why $0! = 1$ that is more than just an ad hoc definition. So, to recap, the $Gamma$ function is a function that is defined and continuous (we won't prove continuity, but trust me) for $x > -1$, and is equal to $(n-1)!$ when x is any natural number.

Now, what does the Gamma function have to do with Laplace transforms? I'm glad you asked. If we want to take the Laplace transform of a function of the form t^a with $a > -1$ we want to calculate the integral:

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a dt$$

where, if we make the substitution $u = st$ we get:

$$\mathcal{L}(t^a) = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a du = \frac{\Gamma(a+1)}{s^{a+1}}$$

In particular, if a is a natural number n then we get:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}.$$

which is consistent with our results for t^2 and 1.

Now, just using the definition of the Laplace transform and some calculus we can calculate the following relations as well, some of which are done for you in the textbook. The textbook also has a useful table of Laplace transforms on page 446.

$$\begin{aligned}
\mathcal{L}(e^{at}) &= \frac{1}{s-a}, s \geq a \\
\mathcal{L}(\cos kt) &= \frac{s}{s^2+k^2} \\
\mathcal{L}(\sin kt) &= \frac{k}{s^2+k^2} \\
\mathcal{L}(\cosh kt) &= \frac{s}{s^2-k^2} \\
\mathcal{L}(\sinh kt) &= \frac{k}{s^2-k^2}
\end{aligned}$$

As I said, all of these can be calculated directly from the definition without too much trouble.

1.3 Important Properties of the Laplace Transform

Here, we'll state without proof some important properties of the Laplace transform. If you're interested in the proofs they're either in your textbook, or in references provided by your textbook.

First, we say that a function $f(t)$ is of *exponential order* as $t \to \infty$ if there exist nonnegative constants $M, c,$ and T such that:

$$|f(t)| \leq Me^{ct} \text{ for } t \geq T.$$

In other words, eventually it's bounded by some exponential function with a lienar argument.

So, for example, every polynomial is of exponential order, while the function e^{t^2} isn't.

Theorem - If the function f is piecewise continuous for $t \geq 0$ and is of exponential order as $t \rightarrow \infty$, then its Laplace transform $F(s) = \mathcal{L}(f(t))$ exists. More precisely, if f is piecewise continuous and satisfies the condition defined above, then $F(s)$ exists for all $s > c$.

Corollary - A result you may have guessed from our examples so far, but a result that comes out from the proof of our theorem is that when the function $f(t)$ satisfies the requirements in our theorem then:

$$\lim_{s \rightarrow \infty} F(s) = 0.$$

Finally, we state a *very* important theorem, especially for what we're going to be doing with Laplace transforms.

Theorem - Suppose that the functions $f(t)$ and $g(t)$ are of exponential order and piecewise continuous, to that their Laplace transforms both exist. If $F(s) = G(s)$ for all $s > c$ (for some c), then $f(t) = g(t)$ wherever on $[0, \infty)$ both f and g are continuous.

This property is extremely important, because what it says is that if we're given a Laplace transform $F(s)$ we can make sense of what we mean by the inverse transform. That is, the "unique" (except for isolated points) function $f(t)$ whose Laplace transform is $F(s)$.

So, for example, if we were told that $F(s) = \frac{1}{s+2}$ then according to our earlier table we could figure out that $f(t) = e^{-2t}$.

2 Transformation of Initial Value Problems

We'll now discuss how we can use Laplace transforms and inverse Laplace transforms to solve initial value problems. In other words, Laplace transforms are going to allow us to take differential equations, and turn them into algebraic equations that we can then solve using algebraic methods to figure out our solution to our differential equation. It's pretty slick.

Before we get into how we do this, we need to establish one *very* important property of the Laplace transform.

Theorem - Suppose that the function $f(t)$ is continuous and piecewise smooth (which means smooth except at isolated points) for $t \geq 0$ and is of exponential order as $t \rightarrow \infty$. Then $\mathcal{L}(f'(t))$ exists for some $s > c$, and

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0) = sF(s) - f(0)$$

and in fact by induction:

$$\mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{n-2}(0) - f^{n-1}(0).$$

Now, this is huge! What it means is that not only can we take Laplace transforms of functions, we can take Laplace transforms of linear differential equations(!) and use these Laplace transforms to find our solutions to the differential equations.

Example - Use Laplace transforms to solve the initial value problem:

$$x'' + 9x = 0; x(0) = 3; x'(0) = 4.$$

If we take the Laplace transform of the differential equation we get:

$$\mathcal{L}(x'' + 9x) = s^2X(s) - sx(0) - x'(0) + 9X(s) = 0$$

where, if we use our initial conditions $x(0) = 3$ and $x'(0) = 4$ and solve for $X(s)$ we get:

$$X(s) = \frac{3s + 4}{s^2 + 9} = 3\frac{s}{s^2 + 9} + 4\frac{1}{s^2 + 9}$$

which, if we look at our table of Laplace transforms, is the Laplace transform of the function:

$$x(t) = 3\cos 3t + \tfrac{4}{3}\sin 3t.$$

Which is the solution to our initial value problem! Let's see that again.

Example - Find the solution of the initial value problem using Laplace transforms:

$$x'' + 8x' + 15x = 0; x(0) = 2; x'(0) = 3$$

If we take the Laplace transform of this relation we get:

$$s^2X(s) - sx(0) - x'(0) + 8sX(s) - 8x(0) + 15X(s) = 0$$

where if we plug in our initial conditions and solve for $X(s)$ we get:

$$X(s) = \frac{2s + 19}{s^2 + 8s + 15}.$$

Now, if we do a partial fraction decomposition, noting that the denominator factors as $(s + 5)(s + 3)$ then we get:

$$X(s) = \frac{-\frac{9}{2}}{s + 5} + \frac{\frac{13}{2}}{s + 3}.$$

In this form the inverse Laplace transform becomes obvious:

$$x(t) = -\tfrac{3}{2}e^{-5t} + \tfrac{7}{2}e^{-3t}.$$

Now, just as with differentiation, we have a relation between the Laplace transform of a function and the Laplace transform of the integral of the function.

Theorem - If $f(t)$ is a piecewise continuous function for $t \geq 0$ and satisfies the condition of exponential order, then

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}\mathcal{L}(f(t)) = \frac{F(s)}{s}$$

for $s > c$. Equivalently,

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)d\tau.$$

Now, these differentiation and integration rules can be exploited to make the calculation of some Laplace transforms much easier.

Example - Find $\mathcal{L}(t \sin kt)$.

This becomes much easier if we first differentiate:

$$f'(t) = \sin kt + kt \cos kt$$

and we note that $f(0) = f'(0) = 0$. If we then differentiate again we get:

$$f''(t) = 2k \cos kt - k^2 t \sin kt.$$

If we note that the laplace transform of $f''(t)$ is just $s^2 F(s)$ we get the relation:

$$s^2 F(s) = \frac{2ks}{s^2 + k^2} - k^2 F(s).$$

If we solve this for $F(s)$ we get:

$$\mathcal{L}(t \sin kt) = F(s) = \frac{2ks}{(s^2 + k^2)^2}.$$

We note that this is much easier than actually evaluating the integral:

$$\mathcal{L}(t \sin kt) = \int_0^\infty t e^{-st} \sin kt dt.$$