# Math 2280 - Lecture 14 

Dylan Zwick

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If we think back to calculus II we'll remember that one of the most important things we learned about in the second half of the class were Taylor series. A Taylor series was a way of expressing a function as an "infinite polynomial". The Taylor series (or in this case the Maclaurin series, which is just the Taylor series expanded around the point $x=0$ ) for some common functions are:

$$
\begin{aligned}
& \sin x=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\cdots \\
& \cos x=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\cdots \\
& e^{x}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+\cdots
\end{aligned}
$$

Now, we're going to use this conception of an exponential function to define what it means to take a "matrix exponential", which is a matrixvalued function of the form:

$$
e^{\mathbf{A}}
$$

where $\mathbf{A}$ is a constant square matrix. Now, it may be that you've never seen this before, and it's not immediately clear what this means. How do you take something to a matrix power? Well, the way that we define this exponential is in terms of an infinite series:

$$
e^{\mathbf{A}}=I+\mathbf{A}+\frac{\mathbf{A}^{2}}{2!}+\frac{\mathbf{A}^{3}}{3!}+\cdots
$$

Now, we'll say that the rigor police are off the beat here and won't go through a proof that this series always converges, but it's true that this series always converges to some constant matrix, where we view convergence of a series of matrices in terms of convergence of their individual entries.

Now, as we're going to see, matrix exponentials have quite a bit to do with finding solutions to first-order linear homogeneous systems with constant coefficient matrices.

## 1 Fundamental Matrices for Systems of ODEs

To review, a solution to a system of ODEs satisfies the relation:

$$
\mathbf{x}^{\prime}=\mathbf{A x}
$$

If we have $n$ linearly independent solutions $\mathbf{x}_{1}, \ldots, \mathbf{x}_{n}$ then any solution to our system can be written in the form:

$$
\mathbf{x}=c_{1} \mathbf{x}_{1}+c_{2} \mathbf{x}_{2}+\cdots+c_{n} \mathbf{x}_{n}
$$

From these linearly independent solutions we can construct a matrix $\Phi(t)$ :

$$
\Phi(t)=\left(\begin{array}{llll}
\mathbf{x}_{1} & \mathbf{x}_{2} & \cdots & \mathbf{x}_{n}
\end{array}\right)
$$

which is called a fundamental matrix for the system defined by the constant matrix A. Now, our statement that we can write any solution $\mathbf{x}$ as a linear combination of our linearly independent solutions can be written in matrix form as:

$$
\mathbf{x}=\Phi(t) \mathbf{c}
$$

where $\mathbf{c}$ is a constant vector given by:

$$
\mathbf{c}=\left(\begin{array}{c}
c_{1} \\
\vdots \\
c_{n}
\end{array}\right)
$$

This fundamental matrix $\Phi(t)$ is not unique, and if $\Psi(t)$ is another fundamental matrix then we have:

$$
\Phi(t)=\left(\begin{array}{cccc}
\tilde{\mathbf{x}}_{1} & \tilde{\mathbf{x}}_{2} & \cdots & \tilde{\mathbf{x}}_{n}
\end{array}\right) .
$$

where the $\tilde{\mathbf{x}}_{i}$ are linearly independent solutions to our system of ODEs. Now, each of these solutions can in turn be represented as a linear combination of our original $\mathbf{x}_{i}$ :

$$
\begin{gathered}
\tilde{\mathbf{x}}_{1}=c_{11} \mathbf{x}_{1}+c_{12} \mathbf{x}_{2}+\cdots+c_{1 n} \mathbf{x}_{n} \\
\tilde{\mathbf{x}}_{2}=c_{21} \mathbf{x}_{1}+c_{22} \mathbf{x}_{2}+\cdots+c_{2 n} \mathbf{x}_{n} \\
\vdots \\
\tilde{\mathbf{x}}_{n}=c_{n 1} \mathbf{x}_{1}+c_{n 2} \mathbf{x}_{2}+\cdots+c_{n n} \mathbf{x}_{n}
\end{gathered}
$$

or, in matrix terminology:

$$
\Psi(t)=\Phi(t) \mathbf{C}
$$

where

$$
\mathbf{C}=\left(\begin{array}{cccc}
c_{11} & c_{21} & \cdots & c_{n 1} \\
c_{12} & c_{22} & \cdots & c_{n 2} \\
\vdots & \vdots & \ddots & \vdots \\
c_{1 n} & c_{2 n} & \cdots & c_{n n}
\end{array}\right)
$$

If we have an initial condition:

$$
\mathbf{x}(0)=\mathbf{x}_{0}
$$

then we get for our solution:

$$
\mathbf{x}_{0}=\Psi(0) \mathbf{c}
$$

where $\mathbf{c}$ is a constant vector given by:

$$
\mathbf{c}=\Psi(0)^{-1} \mathbf{x}_{0}
$$

and we note that $\Psi(0)^{-1}$ makes sense as $\Psi(t)$ is nonsingular by definition.

Combining these results we get:

$$
\mathbf{x}(t)=\Psi(t) \Psi(0)^{-1} \mathbf{x}_{0} .
$$

Example Problem 5.5.2 from the textbook. The problem asks us to find a fundamental matrix for the given system, and then find a solution satisfying the given initial condition.

$$
\mathbf{x}^{\prime}=\left(\begin{array}{cc}
2 & -1 \\
-4 & 2
\end{array}\right) \mathbf{x}
$$

with initial condition

$$
\mathbf{x}(0)=\binom{2}{-1}
$$

To find a fundamental matrix, we need to find two linearly independent solutions to the system. We can get these using the eigenvalue methods from section 5.2. The characteristic equation for our matrix is:

$$
\left|\begin{array}{cc}
2-\lambda & -1 \\
-4 & 2-\lambda
\end{array}\right|=(2-\lambda)^{2}-4=\lambda(\lambda-4)
$$

So, we have two eigenvalues $\lambda=\{0,4\}$. For these two eigenvalues we get the eigenvectors below.

For $\lambda=0$ we have:

$$
\left(\begin{array}{cc}
2 & -1 \\
-4 & 2
\end{array}\right)\binom{a}{b}=\binom{0}{0}
$$

gives us an eigenvector:

$$
\mathbf{v}_{1}=\binom{1}{2}
$$

While for $\lambda=4$ we have:

$$
\left(\begin{array}{ll}
-2 & -1 \\
-4 & -2
\end{array}\right)\binom{a}{b}=\binom{0}{0}
$$

which gives us the eigenvector:

$$
\mathbf{v}_{2}=\binom{1}{-2}
$$

From this we get the fundamental matrix:

$$
\Phi(t)=\left(\begin{array}{cc}
1 & e^{4 t} \\
2 & -2 e^{4 t}
\end{array}\right)
$$

The initial value of this fundamental matrix is:

$$
\Phi(0)=\left(\begin{array}{cc}
1 & 1 \\
2 & -2
\end{array}\right)
$$

which has inverse:

$$
\Phi(0)^{-1}=\frac{1}{-4}\left(\begin{array}{cc}
-2 & -1 \\
-2 & 1
\end{array}\right)=\left(\begin{array}{cc}
\frac{1}{2} & \frac{1}{4} \\
\frac{1}{2} & -\frac{1}{4}
\end{array}\right) .
$$

Using this we get the initial constant vector:

$$
\mathbf{c}=\left(\begin{array}{cc}
\frac{1}{2} & \frac{1}{4} \\
\frac{1}{2} & -\frac{1}{4}
\end{array}\right)\binom{2}{-1}=\binom{\frac{3}{4}}{\frac{5}{4}}
$$

Using this we get the final solution:

$$
\begin{gathered}
\mathbf{x}(t)=\Phi(t) \Phi(0)^{-1} \mathbf{x}_{0}= \\
\left(\begin{array}{cc}
1 & e^{4 t} \\
2 & -2 e^{4 t}
\end{array}\right)\binom{\frac{3}{4}}{\frac{5}{4}}=\binom{\frac{3}{4}+\frac{5}{4} e^{4 t}}{\frac{3}{2}-\frac{5}{2} e^{4 t}}
\end{gathered}
$$

Now, this may seem like a more difficult way of figuring out the same solutions we'd figured out using other methods and, well, that's because it is. However, the major advantage to this method is that it makes it very easy to switch around our initial conditions. Once we've found $\Phi(t)$ and $\Phi(0)^{-1}$, it becomes very easy to find the solution for any given initial conditions $\mathbf{x}_{0}$ just using the relation $\mathbf{x}(t)=\Phi(t) \Phi(0)^{-1} \mathbf{x}_{0}$.

## 2 Matrix Exponentials and ODEs

Right about now you might be asking why we discussed matrix exponentials at the beginning of this lecture. We haven't used them yet. Well, now is when we're going to talk about them, and we'll see they are intimately connected to fundamental matrices.

First, let's go over some important properties of matrix exponentials. If $\mathbf{A}$ is a diagonal matrix:

$$
\mathbf{A}=\left(\begin{array}{lll}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{array}\right)
$$

then if we exponentiate this matrix we get:

$$
e^{\mathbf{A}}=\left(\begin{array}{ccc}
e^{a} & 0 & 0 \\
0 & e^{b} & 0 \\
0 & 0 & e^{c}
\end{array}\right)
$$

Of course the same idea works, mutatis mutandis, for a diagonal matrix of any size.

Also, if two matrices commute, so $\mathbf{A B}=\mathbf{B A}$, then:

$$
e^{\mathbf{A}+\mathbf{B}}=e^{\mathbf{A}} e^{\mathbf{B}} .
$$

Note this this is not necessarily true if A and B do not commute.
Finally, if we have the matrix exponential:

$$
e^{\mathbf{A} t}=\mathbf{I}+\mathbf{A} t+\frac{\mathbf{A}^{2}}{2!} t^{2}+\frac{\mathbf{A}^{3}}{3!} t^{3}+\cdots
$$

then we can differentiate these term by term (again, trust me, the rigor police are off duty here...) and we get:

$$
\frac{d e^{\mathbf{A} t}}{d t}=\mathbf{A}+\mathbf{A}^{2} t+\frac{\mathbf{A}^{3}}{2!} t^{2}+\cdots=\mathbf{A} e^{\mathbf{A} t}
$$

Well, this is very interesting. What this is saying is that the matrix $e^{\mathbf{A t} t}$ satisfies the matrix differential equation:

$$
\mathbf{X}^{\prime}=\mathbf{A} \mathbf{X}
$$

where $\mathbf{X}$ is a square matrix the same size as $\mathbf{A}$, and each column of $\mathbf{x}$ satisfies $\mathbf{x}_{i}^{\prime}=\mathbf{A} \mathbf{x}_{i}$. Now, as $e^{\mathbf{A} t}$ is nonsingular, each of its columns must be linearly independent, and so $e^{\mathbf{A} t}$ is a fundamental matrix for $\mathbf{A}$ !

If we note finally that $e^{\mathbf{A 0}}=\mathbf{I}=\mathbf{I}^{-1}$ then we get the relation:

$$
\mathbf{x}(t)=e^{\mathbf{A} t} \mathbf{x}_{0} .
$$

where $\mathbf{x}(t)$ is a solution to the system of ODEs:

$$
\mathbf{x}^{\prime}(t)=\mathbf{A} \mathbf{x}(t)
$$

with initial condition:

$$
\mathbf{x}(0)=\mathbf{x}_{0} .
$$

Example Solve the system of differential equations:

$$
\mathbf{x}^{\prime}=\left(\begin{array}{lll}
2 & 3 & 4 \\
0 & 2 & 6 \\
0 & 0 & 2
\end{array}\right) \mathbf{x}
$$

with initial conditions:

$$
\mathbf{x}(0)=\left(\begin{array}{c}
19 \\
29 \\
39
\end{array}\right)
$$

To solve this we note that:

$$
\left(\begin{array}{lll}
2 & 3 & 4 \\
0 & 2 & 6 \\
0 & 0 & 2
\end{array}\right)=\left(\begin{array}{lll}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{array}\right)+\left(\begin{array}{lll}
0 & 3 & 4 \\
0 & 0 & 6 \\
0 & 0 & 0
\end{array}\right)
$$

where we call the first matrix in the sum on the right $\mathbf{B}$ and the second matrix C.

Now, we note that:

$$
\mathbf{C}^{2}=\left(\begin{array}{lll}
0 & 3 & 4 \\
0 & 0 & 6 \\
0 & 0 & 0
\end{array}\right)\left(\begin{array}{lll}
0 & 3 & 4 \\
0 & 0 & 6 \\
0 & 0 & 0
\end{array}\right)=\left(\begin{array}{ccc}
0 & 0 & 18 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

and

$$
\mathbf{C}^{3}=\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

This is an example of a matrix that is nilpotent. A matrix $\mathbf{M}$ is nilpotent if $\mathbf{M}^{k}=\mathbf{0}$ for some positive integer $k$. Calculating the matrix exponentials of nilpotent matrices is very easy. It's kind of like calculating Taylor series for polynomials.

Now,

$$
e^{\mathrm{C} t}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)+\left(\begin{array}{lll}
0 & 3 & 4 \\
0 & 0 & 6 \\
0 & 0 & 0
\end{array}\right) t+\frac{1}{2}\left(\begin{array}{ccc}
0 & 0 & 18 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

And so

$$
e^{\mathrm{C} t}=\left(\begin{array}{ccc}
1 & 3 t & 4 t+9 t^{2} \\
0 & 1 & 6 t \\
0 & 0 & 1
\end{array}\right)
$$

while for our matrix $\mathbf{B}$ the matrix exponential is just:

$$
e^{\mathbf{B} t}=\left(\begin{array}{ccc}
e^{2 t} & 0 & 0 \\
0 & e^{2 t} & 0 \\
0 & 0 & e^{2 t}
\end{array}\right)
$$

So, given that $\mathbf{B C}=\mathbf{C B}$ (which is very important for this example) we get:

$$
e^{\mathbf{A} t}=e^{\mathbf{B} t} e^{\mathbf{C} t}=\left(\begin{array}{ccc}
e^{2 t} & 3 t e^{2 t} & \left(4 t+9 t^{2}\right) e^{2 t} \\
0 & e^{2 t} & 6 t e^{2 t} \\
0 & 0 & e^{2 t}
\end{array}\right)
$$

Therefore, the solution to our differential equation is:

$$
\begin{aligned}
\mathbf{x}(t)=e^{\mathbf{A} t} \mathbf{x}_{0} & =\left(\begin{array}{ccc}
e^{2 t} & 3 t e^{2 t} & \left(4 t+9 t^{2}\right) e^{2 t} \\
0 & e^{2 t} & 6 t e^{2 t} \\
0 & 0 & e^{2 t}
\end{array}\right)\left(\begin{array}{l}
19 \\
29 \\
39
\end{array}\right) \\
& =\left(\begin{array}{c}
19+243 t+351 t^{2} \\
29+234 t \\
39
\end{array}\right) e^{2 t} .
\end{aligned}
$$

## 3 Nonhomogeneous Linear Systems

So far we've almost exclusively focused on the system of ODEs:

$$
\mathbf{x}^{\prime}=\mathbf{A x}
$$

However, this is just the homogeneous case, which is pretty much the easiest possible case! In general, we want to be able to solve a system of ODEs of the form:

$$
\mathbf{x}^{\prime}=\mathbf{A} \mathbf{x}+\mathbf{f}(t)
$$

for any value of $\mathbf{f}(t)$. How do we do this? Well, the same way we did it last time. We guess! In particular, if our function $\mathbf{f}(t)$ is a combination (sums and products) of polynomials, exponentials, cosines, and sines, then we use the matrix analogue of the method of undetermined coefficients. This method is easiest explained in the context of an example.

### 3.1 The Method of Undetermined Coefficients

Example - This is a modification of problem 5.6 .3 from the textbook. We want to find a particular solution to the ODEs:

$$
\mathbf{x}^{\prime}=\left(\begin{array}{ll}
3 & 4 \\
3 & 2
\end{array}\right) \mathbf{x}+\binom{0}{t^{2}}
$$

We note that our function $\mathbf{f}(t)$ is a second-order polynomial with constant matrix coefficients, so we guess that our particular solution will also be a second-order polynomial with constant matrix coefficients:

$$
\mathbf{x}_{p}=\mathbf{a}+\mathbf{b} t+\mathbf{c} t^{2}
$$

If we make this guess then we have the relations:

$$
\mathbf{x}_{p}^{\prime}=\left(\begin{array}{ll}
3 & 4 \\
3 & 2
\end{array}\right) \mathbf{x}_{p}+\binom{0}{t^{2}} .
$$

Which in component form gives us:

$$
\binom{b_{1}+2 c_{1} t}{b_{2}+2 c_{2} t}=\binom{\left(3 a_{1}+4 a_{2}\right)+\left(3 b_{1}+4 b_{2}\right) t+\left(3 c_{1}+4 c_{2}\right) t^{2}}{\left(3 a_{1}+2 a_{2}\right)+\left(3 b_{1}+2 b_{2}\right) t+\left(3 c_{1}+2 c_{2}+1\right) t^{2}}
$$

Now, if we equate all the coefficients of $t$ on both sides for each row and solve for the unknowns we get:

$$
\mathbf{x}_{p}=\frac{1}{756}\binom{-868+840 t-504 t^{2}}{861-882 t+378 t^{2}}
$$

Now, the only issue here is the same issue that comes up when our function $\mathbf{f}(t)$ exhibits duplication with our homogeneous solution. So, for example, if we have the system:

$$
\mathbf{x}^{\prime}=\left(\begin{array}{cc}
4 & 2 \\
3 & -1
\end{array}\right) \mathbf{x}-\binom{15}{4} t e^{-2 t}
$$

then out homogeneous solution is:

$$
\mathbf{x}_{h}(t)=c_{1}\binom{1}{-3} e^{-2 t}+c_{2}\binom{2}{1} e^{5 t}
$$

which exhibits duplication with our preliminary trial solution $\mathbf{x}_{p}(t)=$ $\mathbf{a} t e^{-2 t}+\mathbf{b} e^{-2 t}$. Now, this is where this method differs from the earlier method of undetermined coefficients. Here, we need to multiply by $t$ until we no longer exhibit duplication, but we still have to include all lower terms! So, our new trial solution would be:

$$
\mathbf{x}_{p}(t)=\mathbf{a} t^{2} e^{-2 t}+\mathbf{b} t e^{-2 t}+\mathbf{c} e^{-2 t} .
$$

We could solve this, but it turns out that the method of variation of parameters is actually simpler to use here.

### 3.2 The Method of Variation of Parameters

In matrix form the method of variation of parameters is actually pretty easy to formulate.

Suppose we have a system of ODEs:

$$
\mathbf{x}^{\prime}=\mathbf{P}(t) \mathbf{x}+\mathbf{f}(t)
$$

and we already have a complete set of solutions to the associated homogeneous system, and we therefore have a fundamental matrix $\Phi(t)$. Then we're going to try and write our particular solution as a linear combination of our linearly independent solutions to the homogeneous equation, but instead of being a linear combination involving constant coefficients, the coefficients will be functions. In matrix notation this is:

$$
\mathbf{x}_{p}(t)=\Phi(t) \mathbf{u}(t) .
$$

If we differentiate this we get:

$$
\mathbf{x}_{p}^{\prime}(t)=\Phi(t)^{\prime} \mathbf{u}(t)+\Phi(t) \mathbf{u}^{\prime}(t)
$$

Now, if we use the relations:

$$
\begin{gathered}
\mathbf{x}_{p}^{\prime}(t)=\mathbf{P}(t) \mathbf{x}_{p} \\
\text { and } \\
\Phi^{\prime}(t)=\mathbf{P}(t) \Phi(t)
\end{gathered}
$$

then plugging these relations into our above system and simplifying we get:

$$
\Phi(t) \mathbf{u}^{\prime}(t)=\mathbf{f}(t)
$$

So, if we solve this for $\mathbf{u}(t)$ we get:

$$
\mathbf{u}(t)=\int \Phi(t)^{-1} \mathbf{f}(t)
$$

and therefore

$$
\mathbf{x}_{p}(t)=\Phi(t) \int \Phi(t)^{-1} \mathbf{f}(t) .
$$

Now, if $\mathbf{P}(t)=\mathbf{A}$, where $\mathbf{A}$ is a constant matrix, then we have the fundamental matrix $\Phi(t)=e^{\mathbf{A t}}$, and particular solution given by:

$$
\mathbf{x}_{p}(t)=e^{\mathbf{A} t} \int e^{-\mathbf{A} t} \mathbf{f}(t)
$$

and to solve a particular initial value problem we get:

$$
\mathbf{x}(t)=e^{\mathbf{A} t} \mathbf{x}_{0}+e^{\mathbf{A} t} \int_{0}^{t} e^{-\mathbf{A} s} \mathbf{f}(s) d s
$$

