Math 2280 - Lecture 13

Dylan Zwick

Spring 2009

1 Second Order Systems and Mechanical Applications

The major result of our last lecture is that if we have a first order linear homogeneous system of differential equations with constant coefficients:

$$\begin{array}{c} x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{array}$$

we can write this as a matrix equation:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Note that the a_{ij} terms are constants, and that the matrix $\mathbf{A}$ is a constant matrix. The way that we can solve this differential equation is by "guessing" that our solution is of the form:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} e^{\lambda t}$$

If we make this guess, noting that the derivative of $v_i e^{\lambda t}$ is $\lambda v_i e^{\lambda t}$, we get from our differential equation the relation:

$$\lambda \mathbf{v} e^{\lambda t} = \mathbf{A}\mathbf{v} e^{\lambda t}$$

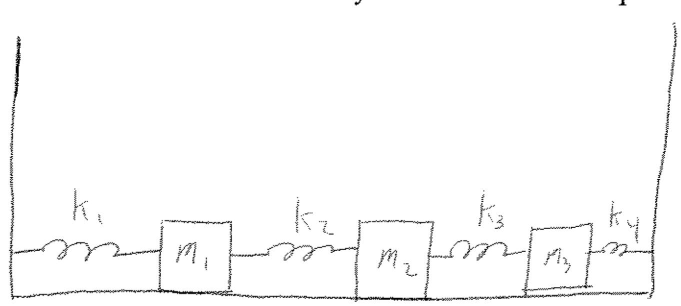
and so if we divide both sides by $e^{\lambda t}$ we get the familiar eigenvalue realtion from linear algebra:

$$\lambda \mathbf{v} = \mathbf{A}\mathbf{v}$$

So, if we make λ an eigenvalue of the matrix $\mathbf{A}$, and $\mathbf{v}$ a corresponding eigenvector, then $\mathbf{v}e^{\lambda t}$ is a solution to our differential equation. If all of our eigenvalues are distinct we can construct a complete linearly independent set of solutions in this manner.

1.1 Second Order Linear Mechanical Systems

Suppose we have a mechanical system like the one pictured below:



A little physical analysis shows the we have the following relations between the positions and accelerations of the three masses:

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' &= -k_2(x_2 - x_1) + k_3(x_3 - x_2) \\ m_3 x_3'' &= -k_3(x_3 - x_2) - k_4 x_3 \end{aligned}$$

which we can rewrite as a matrix equation:

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} =$$
$$\begin{pmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -(k_3+k_4) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

which we can rewrite as:

$$\mathbf{Mx''} = \mathbf{Kx}$$

where the matrix **M** is called the "mass matrix", and the matrix **K** is called the "stiffness matrix". The mass matrix is obviously invertible and easily inverted (just take the reciprocal of the diagonal elements). For example in our case its inverse would be:

$$\mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{pmatrix}$$

and so if we multiply both sides of our differential equation by the inverse of the mass matrix we get:

$$\mathbf{x}'' = \mathbf{Ax}.$$

Look familiar? Well, its very similar to what we discussed last time, except that the derivative on the left is a second derivative, not a first derivative. However, if we make the same guess for our solution, $\mathbf{v}e^{\alpha t}$ (we use α to denote our constant here instead of λ) and try it out then we get the relation:

$$\alpha^2 \mathbf{v} = \mathbf{A}\mathbf{v}$$

So, if $\lambda = \alpha^2$ is a (positive) eigenvalue of our matrix then this gives us two solutions $\mathbf{v}e^{\alpha t}$ and $\mathbf{v}e^{-\alpha t}$ to our differential equation, where $\mathbf{v}$ is an eigenvector of the eigenvalue λ. But, what do we do if our eigenvalue is negative, as is frequently the case for *all* eigenvalues in mechanical systems? Well, in this case if we define $\omega = \sqrt{-\lambda}$ then we get the solution:

$$\mathbf{x}(t) = \mathbf{v}(\cos(\omega t) + i\sin(\omega t))$$

where $\mathbf{v}$ is an eigenvector of λ. Now, both the real part and the imaginary part here are linearly independent solutions, and so we get the two solutions:

$$\mathbf{x}_1(t) = \mathbf{v}\cos(\omega t) \text{ and } \mathbf{x}_2(t) = \mathbf{v}\sin(\omega t).$$

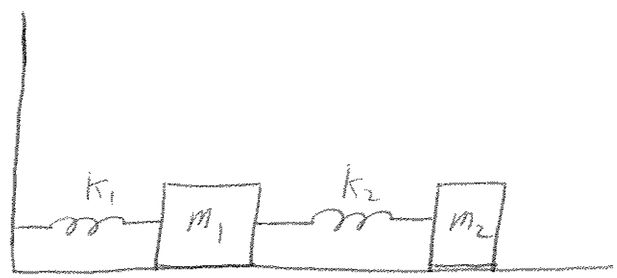
Finally, if the matrix $\mathbf{A}$ has $\lambda = 0$ as an eigenvalue then the corresponding part of our general solution is:

$$\mathbf{x}_0(t) = (a_0 + b_0)\mathbf{v}_0$$

where $\mathbf{v}_0$ is a corresponding eigenvector for $\lambda = 0$. We won't be going over how to derive this today, so just take it as a given.

1.2 A Worked Mechanical Example

Let's examine the dynamics of the example below:



$$m_1 = 2, m_2 = 1$$
$$k_1 = 100, k_2 = 50$$

with corresponding matrix equation:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}'' = \begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix} \mathbf{x}$$

which can be rewritten as:

$$\mathbf{x}'' = \mathbf{A}\mathbf{x}$$

with

$$\mathbf{A} = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}$$

Now, the characteristic polynomial of this matrix is:

$$(-75 - \lambda)(-50 - \lambda) - 25 \times 50 = (\lambda + 25)(\lambda + 100)$$

So, the eigenvalues are $\lambda = \{-25, -100\}$.

This gives us solutions with $\omega = 5$ and $\omega = 10$.

For the $\lambda = -25$ eigenvalue we have the eigenvector equation:

$$\begin{pmatrix} -50 & 25 \\ 50 & -25 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so the eigenvector:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

works. Similarly for the eigenvalue $\lambda = -100$ we get that the vector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector.

The general solution to this system will therefore be:

$$\mathbf{x}(t) = (a_1 \cos 5t + b_1 \sin 5t) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (a_2 \cos 10t + b_2 \sin 10t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where we can see that one component of the solutions oscillates together, while the other component oscillates apart.

1.3 Forced Oscillations and Resonance

Suppose we add a force term to our differential equation:

$$\mathbf{M}\mathbf{x}'' = \mathbf{K}\mathbf{x} + \mathbf{F}$$

where $\mathbf{F}$ is a vector representing the forces on the various masses. Now, if we multiply both sides of this by $\mathbf{M}^{-1}$ we get:

$$\mathbf{x}'' = \mathbf{A}\mathbf{x} + \mathbf{f}$$

where $\mathbf{f} = \mathbf{M}^{-1}\mathbf{F}$.

Now, suppose the force term $\mathbf{f}$ is of the form:

$$\mathbf{f}(t) = \mathbf{F}_0 \cos \omega t.$$

If we guess that a particular solution is of the form:

$$\mathbf{x}_p(t) = \mathbf{c} \cos \omega t$$

then

$$\mathbf{x}_p''(t) = -\omega^2 \mathbf{c} \cos \omega t$$

and so we get the relation for our differential equation:

$$(\mathbf{A} + \omega^2 \mathbf{I})\mathbf{c} = -\mathbf{F}_0.$$

In order for this solution to work we just need to solve the above equation for the vector $\mathbf{c}$, which we can always do providing $(\mathbf{A} + \omega^2 \mathbf{I})$ is nonsingular. If the matrix is singular, which means that $-\omega^2$ is an eigenvalue of our matrix $\mathbf{A}$, then we have a situation called resonance, which can be very bad in physical systems.

To look at this idea in the context of an example, let's return to our first example and add the force term:

$$\mathbf{f}(t) = \begin{pmatrix} 0 \\ 50 \end{pmatrix} \cos \omega t.$$

In this situation our particular solution is $\mathbf{x}_p(t) = \mathbf{c} \cos \omega t$ where the vector $\mathbf{c}$ is the solution to the system:

$$\begin{pmatrix} \omega^2 - 75 & 25 \\ 50 & \omega^2 - 50 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ -50 \end{pmatrix}.$$

Now, if we solve this we get as our values for the components of **c**:

$$c_1 = \frac{1250}{(\omega^2 - 25)(\omega^2 - 100)} \text{ and } c_2 = \frac{50(\omega^2 - 75)}{(\omega^2 - 25)(\omega^2 - 100)}.$$

We can see that everything works out fine here except when $\omega = 5$ or $\omega = 10$, where we would have resonance. If our values of ω are close to these critical values we don't have resonance, but you can see that the numbers c_1 and c_2 could get very large indeed. If you do have a resonance situation then you need to make some modifications to the method we're using here (basically you just multiply your solution by a t), but we won't go into those today.

2 Repeated Eigenvalue Solutions

So far we've only dealt with solutions of the system:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

in which **A** has n distinct eigenvalues if **A** is an $n \times n$ matrix. Now we will deal with the question of what happens if we have less than n distinct eigenvalues, which is what happens if any of the roots of the characteristic polynomial are repeated.

2.1 The Case of an Order 2 Root

Let's start with the case when we have a second order root λ. There are two possible cases here. The first possibility is that we have two distinct (linearly independent) eigenvectors associated with the eigenvalue λ. In this case, all is good, and we just use these two eigenvectors to create two distinct solutions.

Example
Find a general solution of the system:

$$\mathbf{x}' = \begin{pmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{pmatrix} \mathbf{x}$$

The characteristic equation of the matrix $\mathbf{A}$ is:

$$|\mathbf{A} - \lambda \mathbf{I}| = (5-\lambda)(3-\lambda)^2.$$

So, $\mathbf{A}$ has the distinct eigenvalue $\lambda_1 = 5$ and the repeated eigenvalue $\lambda_2 = 3$ of multiplicity 2.

For the eigenvalue $\lambda_1 = 5$ the eigenvector equation is:

$$(\mathbf{A} - 5\mathbf{I})\mathbf{v} = \begin{pmatrix} 4 & 4 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which has as an eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Now, as for the eigenvalue $\lambda_2 = 3$ we have the eigenvector equation:

$$\begin{pmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For this eigenvector equation we have two linearly independent eigenvectors:

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}.$$

and so we have a complete set of linearly independent eigenvectors, and associated solution:

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{5t} + c_2\mathbf{v}_2 e^{3t} + c_3\mathbf{v}_3 e^{3t}$$

Well, that was no problem. Unfortunately, it isn't always the case where we can find two linearly independent eigenvectors for the same eigenvalue. This is the case in the next example.

Example - For the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$$

we have characteristic equation $(\lambda - 4)^2 = 0$, and so we have a root of order 2 at $\lambda = 4$. The corresponding eigenvector equation is:

$$(\mathbf{A} - 4\mathbf{I}) = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We get one eigenvector:

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and that's it! In this situation we call this eigenvalue *defective*, and the *defect* of this eigenvalue is the difference beween the order of the root, and the number of linearly independent eigenvectors. In this case $\lambda = 4$ would be a defective eigenvalue with defect 1.

Now, how does this relate to systems of differential equations, and what do we do in this situation? Well, suppose we have the same matrix **A** as above, and we're given the differential equation:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

For this differential equation we have one solution:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$$

but for a complete set of solutions we'll need another linearly independent solution. How do we get this second solution?

Well, based upon our previous experience we may think that, if $\mathbf{v}_1 e^{\lambda t}$ is a solution to our system, then we can get a second solution of the form $\mathbf{v}_2 t e^{\lambda t}$. Let's try this solution out in our differential equation:

$$\mathbf{v}_2 e^{\lambda t} + \lambda \mathbf{v}_2 t e^{\lambda t} = \mathbf{A}\mathbf{v}_2 t e^{\lambda t}.$$

Well, unfortunately for this to be true it must be true when $t = 0$, which would imply that $\mathbf{v}_2 = \mathbf{0}$, which wouldn't be a linearly independent solution. So, this approach doesn't work.

But wait, there's hope. We don't have to abandon this method entirely. Suppose instead we modify it slightly and replace $\mathbf{v}_2 t$ with $\mathbf{v}_1 t + \mathbf{v}_2$, where $\mathbf{v}_1$ is the one eigenvector that we were able to find. Well, if we plug this into our differential equation we get the relations:

$$\mathbf{v}_1 e^{\lambda t} + \lambda \mathbf{v}_1 t e^{\lambda t} + \lambda \mathbf{v}_2 e^{\lambda t} = \mathbf{A}\mathbf{v}_1 t e^{\lambda t} + \mathbf{A}\mathbf{v}_2 e^{\lambda t}.$$

If we equate the $e^{\lambda t}$ terms and the $te^{\lambda t}$ terms we get the two relations:

$$\begin{gathered} (\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_1 = 0 \\ \text{and} \\ (\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1. \end{gathered}$$

Now, if we think about this, what does this mean. It means that if we have a defective eigenvalue with defect 1, we can find two linearly independent solutions by simply finding a solution to the equation $(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{v}_2 = \mathbf{0}$ such that $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$. If we can find such a vector $\mathbf{v}_2$, then we can construct two linearly independent solutions:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}$$
and
$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t}$$

where $\mathbf{v}_1$ is the non-zero vector given by $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2$.

For our particular situation we have:

$$(\mathbf{A} - 4\mathbf{I})^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, *any* non-zero vector could potentially work as our vector $\mathbf{v}_2$. If we try

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then we get:

$$\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

So, our linearly independent solutions are:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{4t} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^{4t},$$
and
$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{4t} = \begin{pmatrix} -3t + 1 \\ 3t \end{pmatrix} e^{4t}.$$

with corresponding general solution:

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t).$$

We note that our eigenvector $\mathbf{v}_1$ is not our original eigenvector, but is a multiple of it. That's fine.

2.2 The General Case

The vector $\mathbf{v}_2$ above is an example of something called a *generalized eigenvector*. If λ is an eigenvalue of the matrix $\mathbf{A}$, then a *rank r generalized eigenvector* associated with λ is a vector $\mathbf{v}$ such that:

$$(\mathbf{A} - \lambda\mathbf{I})^r\mathbf{v} = \mathbf{0} \text{ but } (\mathbf{A} - \lambda\mathbf{I})^{r-1}\mathbf{v} \neq \mathbf{0}.$$

We note that a rank 1 generalized eigenvector is just our standard eigenvector, where we treat a matrix raised to the power 0 as the identity matrix.

We define a *length k chain of generalized eigenvectors based on the eigenvector $\boldsymbol{v}_1$* as a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of k generalized eigenvectors such that

$$\begin{gathered}
(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_k = \mathbf{v}_{k-1} \\
(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{k-1} = \mathbf{v}_{k-2} \\
\vdots \\
(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1.
\end{gathered}$$

Now, a fundamental theorem of linear algebra states that every $n \times n$ matrix $\mathbf{A}$ has n linearly independent generalized eigenvectors. These n generalized eigenvectors may be arranged in chains, with the sum of the lengths of the chains associated with a given eigenvalue λ equal to the multiplicity of λ (a.k.a. the order of the root in the characteristic equation). However, the structure of these chains can be quite complicated.

The idea behind how we build our solutions is that we calculate the defect d of our eigenvalue λ and we figure out a solution to the equation:

$$(\mathbf{A} - \lambda\mathbf{I})^{d+1}\mathbf{u} = \mathbf{0}.$$

We then successively multiply by the matrix $(\mathbf{A} - \lambda\mathbf{I})$ until the zero vector is obtained. The sequence gives us a chain of generalized eigenvectors, and from these we build up solutions as follows (assuming there are k generalized eigenvectors in the chain):

$$
\begin{aligned}
\mathbf{x}_1(t) &= \mathbf{v}_1 e^{\lambda t} \\
\mathbf{x}_2(t) &= (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t} \\
\mathbf{x}_3(t) &= (\frac{1}{2}\mathbf{v}_1 t^2 + \mathbf{v}_2 t + \mathbf{v}_3) e^{\lambda t} \\
&\vdots \\
\mathbf{x}_k(t) &= \left(\mathbf{v}_1 \tfrac{t^{k-1}}{(k-1)1} + \cdots + \mathbf{v}_{k-2}\tfrac{t^2}{2!} + \mathbf{v}_{k-1} t + \mathbf{v}_k \right) e^{\lambda t}
\end{aligned}
$$

We then amalgamate all these chains of generalized eigenvectors, and these gives us our complete set of linearly independent solutions. This always works. We note that in all the examples we're doing we're assuming all our eigenvalues are real, but that assumption isn't necessary, and this method works just fine if we have complex eigenvalues, as long as we allow for complex eigenvectors as well.

Example

Find the general solution of the system:

$$
\mathbf{x}' = \begin{pmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{x}
$$

The characteristic equation of the coefficient system is:

$$
|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -\lambda & 1 & 2 \\ -5 & -3-\lambda & -7 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda+1)^3
$$

and so the matrix $\mathbf{A}$ has the eigenvalue $\lambda = -1$ with multiplicity 3. The corresponding eigenvector equation is:

$$
\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

The only eigenvectors that work here are of the form:

$$\begin{pmatrix} a \\ a \\ -a \end{pmatrix}$$

and so the eigenvalue $\lambda = -1$ has defect 2. In order to figure out the generalized eigenvectors, we need to calculate $(\mathbf{A} - \lambda\mathbf{I})^2$ and $(\mathbf{A} - \lambda\mathbf{I})^3$:

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$(\mathbf{A} - \lambda\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, what we want to do is find a vector $\mathbf{v}$ such that $(\mathbf{A} - \lambda\mathbf{I})^3\mathbf{v} = \mathbf{0}$ (not hard to do) but also such that $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v} \neq \mathbf{0}$ (also not hard to do, but perhaps not trivial). Let's try the simplest vector we can think of:

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

If we test this out we get:

$$(\mathbf{A} - \lambda\mathbf{I})^3\mathbf{v} = \mathbf{0} \text{ (obviously)}$$
$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v} = \begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}$$

So, all is good here, and we can use this other vector to get our final generalized eigenvector:

$$\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$$

Now, using these three generalized eigenvectors we recover our three linearly independent solutions:

$$\mathbf{x}_1(t) = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} e^{-t}$$

$$\mathbf{x}_2(t) = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} e^{-t}$$

$$\mathbf{x}_3(t) = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} \frac{1}{2}t^2 e^{-t} + \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t}$$

and so our general solution will be:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3$$

where the constants c_1, c_2 and c_3 are determined by initial conditions.