Math 2280 - Lecture 12

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1 A Review of Linear Algebra

First note that we'll be skipping section 4.3. I think we've had enough numerical methods for now. So, we're moving on to chapter 5, which concerns linear systems of differential equations, and is when we're going to start really dealing with the intersection of linear algebra and differential equations. So, before we start, we'll begin with a brief refresher of the basics of linear algebra that you'll be expected to know. Please note that this is expected to be a *review*, and you should be familiar with all the review material presented here.

The fundamental object we study in linear algebra is the matrix, which is a rectangular array of numbers:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

This is an $m \times n$ matrix, which means that it has m rows and n columns. We note that the number of rows and number of columns do not have to be the same, although frequently for the matrices we deal with they will be.

1.1 Basic Operations

Addition of matrices if termwise. We note that we can only add two matrices if they are the same size. Multiplying a matrix by a constant multiplies every element in the matrix by that constant.

Examples

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 9 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 8 & 13 \end{pmatrix}$$

$$4 \begin{pmatrix} 2 & 9 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 8 & 36 \\ 20 & 24 \end{pmatrix}$$

Now, matrix addition and scalar multiplication of matrices satisfy the following basic properties:

1. $\mathbf{A} + \mathbf{0} = \mathbf{A}$

2. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

3. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

4. $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

5. $(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$

where here bold faced capitals (majuscules) represent matrices, lower case standard letters represent scalars, and $\mathbf{0}$ represents the zero matrix of the appropriate size, the zero matrix being the matrix all of whose entries are 0.

1.2 Dot Products and Matrix Multiplication

Now, $1 \times n$ matrices are frequently called vectors or column vectors, while $m \times 1$ matrices are frequently called row vectors. We can take the dot product of two vectors (which can be formulated as the product of a row vector with a column vector) by summing the products of the entries on the two vectors. So, for example, if:

$$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix}$$

then $\mathbf{a} \cdot \mathbf{b} = 3 \times 2 + 4 \times 9 + 7 \times 6 = 84$.

Now, just as matrices can be added they can be multiplied. The basic formula for matrix multiplication is:

$$\mathbf{AB}_{ij} = \sum_{r=1}^{k} a_{ik} b_{kj}$$

where the matrix $\mathbf{AB}$ is the matrix $\mathbf{A}$ right multiplied by the matrix $\mathbf{B}$. We note that for this to work out the number of columns in $\mathbf{A}$ must equal the number of rows in $\mathbf{B}$. We can view the entry $\mathbf{AB}_{ij}$ as being the dot product of the ith row of matrix $\mathbf{A}$ with the jth column of matrix $\mathbf{B}$.

Note that matrix multiplication is in general *not* commutative, so it's not always the case that $\mathbf{AB} = \mathbf{BA}$ even when this makes sense (i.e. they're both square).

1.3 Inverses

Now, for square matrices we can talk about inverses. For a square matrix $\mathbf{A}$, its inverse (if it exists) is the unique matrix $\mathbf{A}^{-1}$ such that:

$$\mathbf{AA}^{-1} = \mathbf{I}$$

where the matrix $\mathbf{I}$ is the *identity matrix* that is a square matrix with the entry 1 down the diagonal and the entry 0 everywhere else. The identity matrix is called this because if you multiply it with any other appropriate (same size) matrix you just get back that matrix. We again stress that these concepts only make sense for square matrices.

1.4 Determinants

For a 2×2 matrix the *determinant* is defined as:

$$det(\mathbf{A}) = det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For higher order determinants they can be calculated by expansion. So, for example:

$$\begin{vmatrix} 3 & 2 & 1 \\ 5 & 6 & 2 \\ 0 & 1 & 4 \end{vmatrix} = 3 \begin{vmatrix} 6 & 2 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 5 & 2 \\ 0 & 4 \end{vmatrix} + 1 \begin{vmatrix} 5 & 6 \\ 0 & 1 \end{vmatrix} = 31$$

We note that we'd get the same value if we chose to expand along any row or column.

Now, a big (in some sense *the* big) theorem of linear algebra states that a squre matrix is invertible if and only if it has a non-zero determinant. The concept of determinant as such doesn't make sense for non-square matrices.

1.5 Matrix-Valued Functions

The entries of a vector of matrix for that matter don't have to be constants, and they can even be functions. So, for example, a vector-valued function is a vector all of whose components are functions:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

where the $x_i(t)$ are standard (scalar-valued) functions.

In a similar way we can define matrix-valued functions:

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

We can make sense of the derivative of vector or matrix valued functions just by defiing it as the derivative of each entry termwise. If we define differentiation of matrix-valued functions this way we recover a form of the product rule:

$$\frac{d(\mathbf{AB})}{dt} = \frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt}.$$

We note that these terms are all matrices, and matrix multiplication is in general *not* commutative, so don't switch these terms around even if they're square and you can!

Example

$$\mathbf{A}(t) = \begin{pmatrix} e^t & t^2 - 2t \\ \cos(4t) & 3^5 + 1 \end{pmatrix}$$

The derivative, $\mathbf{A}'(t)$, of this matrix is:

$$\mathbf{A}'(t) = \begin{pmatrix} e^t & 2t - 2 \\ -4\sin(4t) & 3^t \ln 3 \end{pmatrix}$$

1.6   First-Order Linear Systems

As mentioned in sectin 4.2, if we have a first-order linear system of differential equations:

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t)$$
$$x_2' = p_{21}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t)$$
$$\cdots$$
$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t)$$

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we can express this in a much simpler form as a matrix equation:

$$\frac{d\mathbf{x}}{dt} = \mathbf{Px} + \mathbf{f}$$

where $\mathbf{x}_{i1} = x_i(t)$, $\mathbf{P}_{ij} = p_{ij}(t)$, and $\mathbf{f}_{i1} = f_i(t)$.

A solution to this system of differential equations is a vector valued function $\mathbf{x}$ that satisfies the above differential equation. Now, we call such an equation *homogeneous* if, you guessed it, $\mathbf{f} = \mathbf{0}$.

1.7 Superposition

Suppose we have a homogeneous differential equation in the form above and we have solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then any other vector-valued function of the form:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

is also a solution. This follows immediately from the linearity properties of, well, *linear* algebra.

Now, it turns out (not too surprisingly) that *any* solution of a homogeneous differential equation in the form mentioned above can be written as a linear combination of n *linearly independent* solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

We can determine if n solution vectors are linearly independent by checking their (surprise!) Wronskian:

$$W(\mathbf{x}_1, \ldots, \mathbf{x}_n) = det([\mathbf{x}_1 \cdots \mathbf{x}_n]).$$

Example - First verify that the given vectors are solutions of the given ssytem. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system.

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}_1 = \begin{pmatrix} 2e^t \\ -3e^5 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$$

2 The Eigenvalue Method for Homogeneous Systems

Suppose we have a system of first-order ODEs with *constant* coefficients:

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\quad\vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}
$$

We know that *any* solution (general theory) can be written as the linear combination:

$$
\mathbf{x}(t) = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n
$$

where the $\mathbf{x}_i$ are linearly independent solutions of the system of ODEs. So, what we want to do is figure out how to find these linearly independent solutions.

2.1 The Exponential "Guess"

By analogy with the constant coefficient case for homogeneous ODEs, we can "guess" a solution of the form:

$$
\mathbf{x}(t) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} e^{\lambda t} = \mathbf{v}e^{\lambda t}
$$

where the v_i and λ are appropriate scalar constants.

Now, if we write our system as:

$$
\mathbf{x}' = \mathbf{A}\mathbf{x}
$$

then if $\mathbf{x} = \mathbf{v}e^{\lambda t}$ then we get:

$$\lambda \mathbf{v} e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t}$$
$$\Rightarrow \lambda \mathbf{v} = \mathbf{A}\mathbf{v}$$

which is the "eigenvalue equation" from linear algebra.

2.2   The Eigenvalue Equation

We begin with a theorem from linear algebra. Namely, that $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ for $\mathbf{v} \neq 0$ if and only if $det(\mathbf{A} - \lambda \mathbf{I}) = 0$. This theorem determines the possible values of λ.

In general

$$det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

gives us a "characteristic" nth-order (in λ) polynomial whose roots are the acceptable values of λ.

Well, if we get n *distinct* eigenvalues, as these roots are called, then we get n linearly independent solutions, and we're done. Now, as you might imagine, these solutions may be complex conjugates, a situation we'll discuss today. We'll delay what we do if any of the eigenvalues are repeated until section 5.4.

2.3   All Real Roots

If all the roots are real and distinct, then the problem is as easy as it can be. How this is handled is best seen in the context of an example.

Example - Find the general solution of:

$$\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x}$$

Continued...

2.4 Complex Eigenvalues

Any complex eigenvalue will also have its conjugate as an eigenvalue:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0$$
$$\Rightarrow (\mathbf{A} - \overline{\lambda}\mathbf{I})\overline{\mathbf{v}} = 0$$

So, $\overline{\mathbf{v}}$ is a corresponding eigenvector to the eigenvalue $\overline{\lambda}$. Now, if λ is complex then we have:

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} = \mathbf{v}e^{(p+qi)t} = (\mathbf{a} + \mathbf{b}i)e^{pt}(\cos(qt) + i\sin(qt))$$

which gives us,
$$\mathbf{x}(t) = e^{pt}(\mathbf{a}\cos(qt) - \mathbf{b}\sin(qt)) + ie^{pt}(\mathbf{b}\cos(qt) + \mathbf{a}\sin(qt))$$

Now, as $0 = 0 + i0$, both the real term and complex term here must be a solution to the system of ODEs, and these are the same pair of solutions we'll get from the eigenvalue's conjugate. So, our two linearly independent solutions, arising from the eigenvalue and its conjugate, are the real and imaginary parts above.

Example - Find the solution to the given system of ODEs:

$$x_1' = x_1 - 2x_2$$
$$x_2' = 2x_1 + x_2$$
$$x_1(0) = 0, \, x_2(0) = 4$$

continued...