Math 2280 - Lecture 10

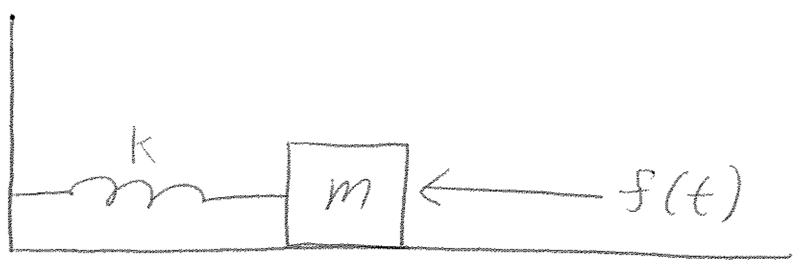
Dylan Zwick

Spring 2009

1 Mechanical Vibrations, Forced oscillations, and Resonance

In this lecture we'll review in greater depth the simple mechanical system we discussed in our last lecture, and discuss some of the consequences of adding a forcing function to the system.

Suppose we have a spring-mass system with an external driving force, pictured schematically below:



Assuming there is no damping, we can model this system by a differential equation of the form:

$$mx'' + kx = f(t)$$

Now, suppose our forcing function is of the form $f(t) = F_0 \cos \omega t$, where $\omega \neq \sqrt{k/m}$ then the method of undetermined coefficients would lead us to guess a particular solution of the form:

$$x(t) = A\cos\omega t + B\sin\omega t.$$

Now, if we plug this guess into our differential equation we get the relation:

$$-Am\omega^2\cos\omega t + Ak\cos\omega t - Bm\omega^2\sin\omega t + Bk\sin\omega t = F_0\cos\omega t$$

which if we solve for the constants A and B we get:

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}$$
$$B = 0.$$

Consequently, our particular solution will be:

$$x_p(t) = \left(\frac{F_0/m}{\omega_0^2 - \omega^2}\right)\cos\omega t.$$

And, in general, our solution will be of the form:

$$x(t) = \left(\frac{F_0/m}{\omega_0^2 - \omega^2}\right)\cos\omega t + c_1\sin\omega_0 t + c_2\cos\omega_0 t.$$

1.1 Beats

If we impose the initial conditions: $x(0) = x'(0) = 0$ then we have:

$$c_1 = 0$$

and

$$c_2 = -\frac{F_0/m}{\omega_0^2 - \omega^2}.$$

So, for our solution we get:

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t)$$

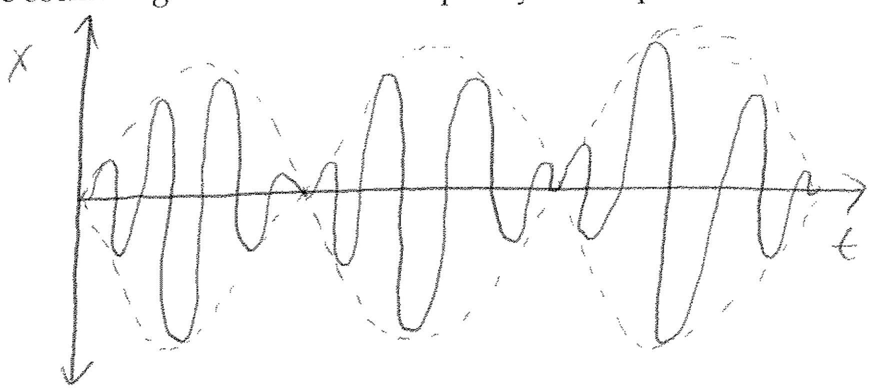
where if we use the relation

$$2 \sin A \cos B = \cos (A - B) - \cos (A + B)$$

we get:

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \left(\frac{(\omega_0 - \omega)}{2} t \right) \sin \left(\frac{(\omega_0 + \omega)}{2} t \right).$$

Now, we have if $\omega_0 \approx \omega$, this solution looks like a higher frequency wave oscillating within a lower frequency envelope:



This is a situation known as beats.

1.2 Resonance

What if $\omega = \omega_0$? Then, for our particular solution we'd guess:

$$x_p(t) = At \cos (\omega_0 t) + Bt \sin (\omega_0 t).$$

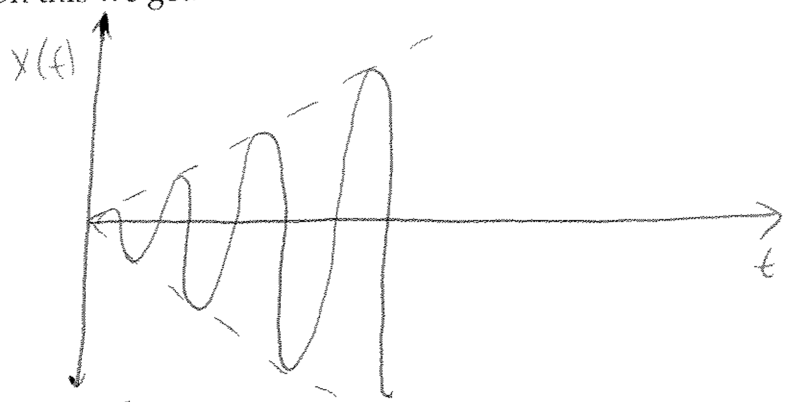
If we make this guess and work it out with the initial conditions $x(0) = x'(0) = 0$ we get:

$$A = 0$$
$$B = \frac{F_0}{2m\omega_0}$$

with corresponding particular solution:

$$x_p(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

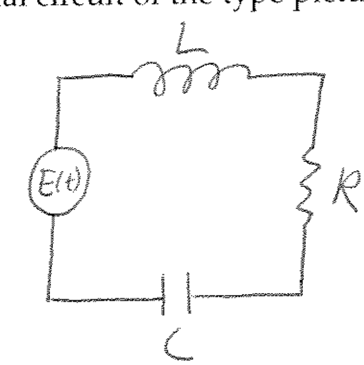
If we graph this we get:



This is a situation known as resonance.

2 Electrical Circuits

For an electrical circuit of the type pictures below:



Kirchoff's second law tells us that the sum of the voltage drops across each component must equal 0:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t).$$

This is a second order linear ODE with constant coefficients! Now, this is the same mathematics as the mechanical system we just studied, and so we'll have the same solutions.

So, for example if $E(t) = E_0 \sin(\omega t)$ then (if we differentiate both sides of the above eqution) we get:

$$LI'' + RI' + \frac{1}{C}I = \omega E_0 \cos \omega t.$$

The complete solution to this will be:

$$y_h = e^{-\frac{Rt}{2L}}\left(c_1 e^{\frac{\sqrt{R^2-4L/C}}{2L}t} + c_2 e^{-\frac{\sqrt{R^2-4L/C}}{2L}t}\right).$$

This gives us a solution for I_{tr}, the *transient* current that will die our exponentially.

Now, the particular solution will give us another term called the steady periodic current. If we run through the math, which is exactly the same as in the mechanical system, we get:

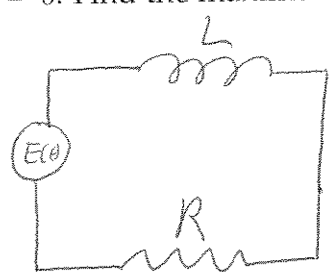
$$I_{sp}(t) = \frac{E_0 \cos \omega t - \alpha}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

where

$$\alpha = \arctan\left(\frac{\omega RC}{1 - LC\omega^2}\right).$$

The quantity in the denominator of our steady periodic current is denoted by the variable Z and is called the *impedence* of the circuit. The term $\omega L - 1/(\omega C)$ is called the *reactance*.

Example - In the circuit below, suppose that $L = 2, R = 40, E(t) = 100e^{-10t}$, and $I(0) = 0$. Find the maximum current for $t \geq 0$.



Our differential equation is:

$$2\frac{dI}{dt} + 40I = 100e^{-10t}$$

which if we solve using the methods from chapter 1 we get:

$$I(t) = 5e^{-10t} - 5e^{-20t}.$$

Using calculus to find the maximum of this function we find that the maximum current occurs when $t = \ln 2$ and has a value of $1.25A$, where A is the unit of Amperes.