Math 2280 - Final Exam

University of Utah

Spring 2009

Name: Solutions

Laplace Transforms You May Need

Definition

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Eigenvalue Rules for Critical Points

$\lambda_1 < \lambda_2 < 0$ Stable improper node

$\lambda_1 = \lambda_2 < 0$ Stable node or spiral point

$\lambda_1 < 0 < \lambda_2$ Unstable saddle point

$\lambda_1 = \lambda_2 > 0$ Unstable node or spiral point

$\lambda_1 > \lambda_2 > 0$ Unstable improper node

$\lambda_2, \lambda_2 = a \pm bi, (a < 0)$ Stable spiral point

$\lambda_1, \lambda_2 = a \pm bi, (a > 0)$ Unstable spiral point

$\lambda_1, \lambda_2 = \pm bi$ Stable or unstable, center or spiral point

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

Basic Definitions (5 points)

Circle or state the correct answer for the questions about the following differential equation:

$$x^2y'' - \sin(x)y' + y^3 = e^{2x}$$

(1 point) The differential equation is: Linear Nonlinear

(1 points) The order of the differential equation is: 2

For the differential equation:

$$(x^4 - x)y^{(3)} + 2xe^xy' - 3y = \sqrt{x - \cos(x)}$$

(1 point) The differential equation is: Linear Nonlinear

(1 point) The order of the differential equation is: 3

(1 point) The corresponding homogeneous equation is:

$$(x^4 - x)y^{(3)} + 2xe^xy' - 3y = 0$$

Separable Equations (5 points)

Find the general solution to the differential equation:

$$\frac{dy}{dx} = 3\sqrt{xy}$$

$$\Rightarrow \int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x}\, dx$$

$$\Rightarrow 2\sqrt{y} = 2x^{3/2} + C$$

$$\Rightarrow \sqrt{y} = x^{3/2} + C$$

$$\Rightarrow \boxed{y = \left(x^{3/2} + C\right)^2}$$

Linear First-Order Equations (5 points)

Find the particular solution to the differential equation below with the given value:

$$xy' + 3y = 2x^5;$$
$$y(2) = 1.$$

$$y' + \frac{3}{x} y = 2x^4$$

$$\rho(x) = e^{\int \frac{3}{x} dx} = e^{3\ln|x|} = x^3$$

$$\Rightarrow y'x^3 + 3x^2 y = 2x^7$$

$$\Rightarrow \frac{d}{dx}(x^3 y) = 2x^7$$

$$\Rightarrow x^3 y = \frac{x^8}{4} + C \Rightarrow y = \frac{x^5}{4} + \frac{C}{x^3}$$

$$y(2) = \frac{2^5}{4} + \frac{C}{2^3} = 8 + \frac{C}{8} = 1$$

$$\Rightarrow C = -56$$

$$\boxed{y(x) = \frac{x^5}{4} - \frac{56}{x^3}}$$

Continued...

Higher Order Linear Differential Equations (5 points)

Find the general solution to the linear differential equation:

$$y'' - 3y' + 2y = 0.$$

The characteristic equation is

$$x^2 - 3x + 2 = 0$$

$$\Rightarrow (x-2)(x-1) = 0$$

So, the roots are: $x = 1, 2$.

The general solution is:

$$\boxed{y(x) = c_1 e^x + c_2 e^{2x}}$$

Nonhomogeneous Linear Differential Equations (10 points)

Find the general solution to the differential equation:

$$y^{(3)} + 4y' = 3x - 1.$$

The corresponding homogeneous equation is:

$$y^{(3)} + 4y' = 0$$

with characteristic equation:

$$x^3 + 4x = 0$$

$$\Rightarrow x(x^2+4) = 0$$

The roots are: $x = 0, \pm 2i$

So, the homogeneous solution is:

$$y_h(x) = c_1 e^{0x} + c_2 \cos(2x) + c_3 \sin(2x)$$

$$= c_1 + c_2 \cos(2x) + c_3 \sin(2x)$$

Now, the particular solution is:

$$y_p(x) = Ax^2 + Bx$$

where we note the nonhomogeneous term is $3x-1$, which is a first-order polynomial, but the constant term is not linearly independent of our homogeneous solution, so we need to multiply our "guess" ~~bx~~ by x, giving us:

$$y_p = Ax^2 + Bx$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p^{(3)} = 0$$

$$\Rightarrow \quad 4(2Ax + B) = 3x - 1$$

$$8Ax + 4B = 3x - 1$$

$$B = -\frac{1}{4} \quad A = \frac{3}{8}$$

So, $\quad y_p = \frac{3}{8}x^2 - \frac{1}{4}x$

Our general solution is then:

$$\boxed{y(x) = c_1 + c_2\cos(2x) + c_3\sin(2x) + \frac{3}{8}x^2 - \frac{1}{4}x}$$

Systems of Differential Equations (10 points)

Find the general solution to the system of differential equations:

$$\begin{aligned} x_1' &= 5x_1 + x_2 + 3x_3 \\ x_2' &= x_1 + 7x_2 + x_3 \\ x_3' &= 3x_1 + x_2 + 5x_3 \end{aligned}$$

$$\vec{x}\,' = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 5-\lambda & 1 & 3 \\ 1 & 7-\lambda & 1 \\ 3 & 1 & 5-\lambda \end{vmatrix} = (5-\lambda)\left[(7-\lambda)(5-\lambda)-1\right] - 1\left[1(5-\lambda)-3\right] + 3\left[1(1)-(7-\lambda)(3)\right]$$

$$= (5-\lambda)\left[\lambda^2-12\lambda+34\right] + (\lambda-2) + 3(3\lambda-20)$$

$$= 5\lambda^2 - 60\lambda + 170 - \lambda^3 + 12\lambda^2 - 34\lambda + \lambda - 2 + 9\lambda - 60$$

$$= -\lambda^3 + 17\lambda^2 - 84\lambda + 108$$

$$= (2-\lambda)(\lambda^2 - 15\lambda + 54) = (2-\lambda)(\lambda-9)(\lambda-6)$$

So, the eigenvalues are $\lambda = 2, 6, 9$.

Continued...

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$ is an eigen vector for $\lambda = 2$

$$\begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$ is an eigenvector for $\lambda = 6$.

$$\begin{pmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ is an eigenvector for $\lambda = 9$.

So, the general solution is:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{6t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t}}$$

Continued...

Systems of Differential Equations with Repeated Eigenvalues (5 points)

Find a general ~~solutions~~ solution ~~of~~ to the system of differential equations:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix} \mathbf{x}.$$

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = (\lambda-1)(\lambda-9)+16$$
$$= \lambda^2 - 10\lambda + 25$$
$$= (\lambda-5)^2$$

So, $\lambda = 5$ is the only eigenvalue.

$$(A-5I) = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the "only" eigenvector, up to a constant multiple.

So, we need a length 2 chain:

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1$$
$$(A-\lambda I)\vec{v}_1 = \vec{0}$$

$$\Rightarrow (A-\lambda I)^2 \vec{v}_2 = \vec{0}$$

$$(A-\lambda I)^2 = (A-5I)^2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Continued...

So, any vector $\vec{v}_2$ works so long as $(A-5I)\vec{v}_2 \neq \vec{0}$.

Take $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(A-5I)\vec{v}_2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = \vec{v}_1.$$

So, our solutions are:

$$\vec{x}_1(t) = \vec{v}_1 e^{5t}$$

$$\vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2)e^{5t}$$

Therefore,

$$\boxed{\begin{aligned} \vec{x}(t) &= c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) \\ &= c_1 \begin{pmatrix} -4 \\ 4 \end{pmatrix} e^{5t} + c_2 \left[\begin{pmatrix} -4 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{5t} \end{aligned}}$$

Laplace Transforms (5 points)

Using the definition of the Laplace transform calculate the Laplace transform of the function:

$$f(t) = e^{3t+1}.$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} e^{3t+1}\, dt$$

$$= \int_0^\infty e^{(3-s)t+1}\, dt$$

$$= \left. \frac{e^{(3-s)t+1}}{(3-s)} \right|_0^\infty$$

$$= \boxed{\frac{-e}{3-s}} \quad \text{for } s > 3$$

undefined for $s \leq 3$.

Note: this is, of course, the same as:

$$= \frac{e}{s-3} = e\left(\frac{1}{s-3}\right) \qquad s > 3.$$

Laplace Transforms and Differential Equations (8 points)

Find the particular solution to the differential equation:

$$x'' + 4x = \delta(t) + \delta(t - \pi);$$

$$x(0) = x'(0) = 0.$$

$$\mathcal{L}(x(t)) = X(s)$$

$$\mathcal{L}(x'(t)) = sX(s) - x(0) \quad = sX(s)$$

$$\mathcal{L}(x''(t)) = s^2X(s) - sx(0) - x'(0) = s^2X(s)$$

$$\mathcal{L}(\delta(t)) = 1, \quad \mathcal{L}(\delta(t-\pi)) = e^{-\pi s}$$

$$\Rightarrow \mathcal{L}(x'' + 4x) = \mathcal{L}(\delta(t) + \delta(t-\pi))$$

$$\Rightarrow (s^2+4)X(s) = 1 + e^{-\pi s}$$

$$\Rightarrow \quad X(s) = \frac{1+e^{-\pi s}}{s^2+4} = \frac{1}{s^2+4} + \frac{e^{-\pi s}}{s^2+4}$$

The inverse Laplace transform will be:

$$= \frac{1}{2}\sin(2t) + u(t-\pi)\frac{1}{2}\sin(2(t-\pi))$$

$$= \boxed{\frac{1}{2}\sin(2t)(1+u(t-\pi))}$$ noting $\sin(2t-2\pi) = \sin(2t)$

Continued...

Nonlinear Systems (7 points)

Determine the location of the critical point (x_0, y_0) for the system given below, and classify the critical point as to its type and stability.

$$\frac{dx}{dt} = x + y - 7,$$

$$\frac{dy}{dt} = 3x - y - 5.$$

$$\frac{dx}{dt} = x + y - 7 = 0$$

$$\frac{dy}{dt} = 3x - y - 5 = 0$$

$$\Rightarrow \quad x + y = 7, \quad 3x - y = 5 \quad \Rightarrow 4x = 12 \quad \Rightarrow x = 3, \; y = 4$$

So, $(x_0, y_0) = (3, 4)$ is the only critical point.

The Jacobian is:

$$J(x,y) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \Rightarrow J(3,4) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

So,

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1) - 3 = \lambda^2 - 4$$

$\Rightarrow \lambda = \pm 2$ are the eigenvalues.

Continued...

Two real eigenvalues of opposite sign means (3, 4) is an unstable saddle point.

More Nonlinear Systems (10 points)

For the nonlinear system below, determine all critical points, and classify each according to its type and stability.

$$\frac{dx}{dt} = 3x - x^2 + \frac{1}{2}xy,$$

$$\frac{dy}{dt} = \frac{1}{5}xy - y.$$

$\frac{dx}{dt} = 0 \quad \frac{dy}{dt} = 0$

$\Rightarrow 3x - x^2 + \frac{1}{2}xy = x(3 - x + \frac{1}{2}y) = 0$

$\Rightarrow \frac{1}{5}xy - y = y(\frac{1}{5}x - 1) = 0$

Both are zero if $x = y = 0$.

If $x \neq 0$, then if $y = 0$ we must have $x = 3$.

If $y \neq 0$, then we must have $x = 5$, and if $x = 5$ we must have $y = 4$.

So, $(0,0)$, $(3,0)$, $(5,4)$ are the critical points.

The Jacobian of the system is:

$$J(x,y) = \begin{pmatrix} 3 - 2x + \frac{1}{2}y & \frac{1}{2}x \\ \frac{1}{5}y & \frac{1}{5}x - 1 \end{pmatrix}$$

Continued...

So,

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{vmatrix} 3-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3)$$

So, the eigenvalues are $\lambda = -1$, $\lambda = 3$ which corresponds with an unstable saddle point.

$$J(3,0) = \begin{pmatrix} -3 & \frac{3}{2} \\ 0 & \frac{3}{5}-1 \end{pmatrix} = \begin{pmatrix} -3 & \frac{3}{2} \\ 0 & -\frac{2}{5} \end{pmatrix}$$

$$\begin{vmatrix} -3-\lambda & \frac{3}{2} \\ 0 & -\frac{2}{5}-\lambda \end{vmatrix} = \left(\lambda+\frac{2}{5}\right)(\lambda+3), \quad \Rightarrow \lambda = -\frac{2}{5}, -3$$ are the eigenvalues. So, $(3,0)$ is a stable nodal sink.

$$J(5,4) = \begin{pmatrix} -5 & \frac{5}{2} \\ \frac{4}{5} & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -5-\lambda & \frac{5}{2} \\ \frac{4}{5} & -\lambda \end{vmatrix} = \lambda(\lambda+5) - 2 = \lambda^2 + 5\lambda - 2 = 0$$

$$\Rightarrow \lambda = \frac{-5 \pm \sqrt{5^2 - 4(1)(-2)}}{2} = \frac{-5 \pm \sqrt{25+8}}{2} = \frac{-5 \pm \sqrt{33}}{2}$$

So, two real eigenvalues of opposite sign, therefore $(5,4)$ is an unstable saddle point.

Continued...

So, the critical points are:
$(0,0)$, $(3,0)$, and $(5,4)$, with critical point behavior:

$(0,0) \Rightarrow$ unstable saddle point

$(3,0) \Rightarrow$ stable nodal sink (improper node)

$(5,4) \Rightarrow$ unstable saddle point.

Ordinary, Regular, and Irregular Points (5 points)

Determine if the point $x = 0$ in the following second order differential equation is an ordinary point, a regular singular point, or an irregular singular point.

$$x^3 y'' + 6\sin(x) y' + 6xy = 0.$$

$$\Rightarrow \quad y'' + \frac{6\sin(x)}{x^3} y' + \frac{6}{x^2} y = 0$$

$\lim_{x \to 0} \frac{6}{x^2} = \infty$, which does not exist, so $x = 0$ is a singular point.

$P(x) = \frac{6\sin(x)}{x^3}$, $\Rightarrow p(x) = xP(x) = \frac{6\sin(x)}{x^2}$

$\Rightarrow \lim_{x \to 0} \frac{6\sin(x)}{x^2} = \pm\infty$, which does not exist, so $x = 0$ is an <u>irregular singular point.</u>

Power Series Solutions (10 points)

Find a general solution in powers of x to the differential equation:

$$(x^2+1)y'' + 6xy' + 4y = 0.$$

$$\Rightarrow \quad y'' + \frac{6x}{x^2+1}y' + \frac{4}{x^2+1}y = 0$$

$$\lim_{x\to 0} \frac{6x}{x^2+1} = \cancel{\lim_{x}}\ 0$$

$$\lim_{x\to 0} \frac{4}{x^2+1} = 4$$

So, $x=0$ is an ordinary point.

So, we must find solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\Rightarrow (x^2+1)y'' + 6xy' + 4y = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n n(n-1)x^n + \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} 6c_n n x^n + \sum_{n=0}^{\infty} 4c_n x^n = 0$$

Continued...

$$\Rightarrow \sum_{n=0}^{\infty} \left[c_n n(n-1) + 6c_n n + 4c_n + c_{n+2}(n+2)(n+1) \right] x^n$$

We note that no restriction is placed on c_0 or c_1, which are arbitrary. The recursion relation for the higher order constants is:

$$c_n (n+4)(n+1) + c_{n+2}(n+2)(n+1) = 0$$

$$\Rightarrow \quad c_{n+2} = -\frac{c_n (n+4)}{(n+2)}$$

$$\Rightarrow \quad c_2 = \frac{-4c_0}{2} = -2c_0$$

$$c_3 = -\frac{5c_1}{3}$$

$$c_4 = -\frac{6c_2}{4} = 3c_0$$

$$c_5 = \frac{-7c_3}{5} = \frac{7\times 5}{5\times 3} c_1 = \frac{7}{3} c_1$$

$$c_6 = \frac{-8c_4}{6} = -4c_0$$

$$c_7 = \frac{-9c_5}{7} = -\frac{9}{3} c_1$$

So, we see the pattern:

$$c_{2n} = (-1)^n (n+1) c_0$$

$$c_{2n+1} = \frac{(-1)^n (2n+3) c_1}{3}$$

So, our two solutions are:

$$y_1(x) = c_0 \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n}$$

$$y_2(x) = \frac{c_1 x}{3} \sum_{n=0}^{\infty} (-1)^n (2n+3) x^{2n}$$

(Full credit if you get this far.)

Now, we note: (assuming $|x|<1$)

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{and} \quad \sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$$\Rightarrow \text{ so, } \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

$$y_1(x) = c_0 \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n} = c_0 \left(\sum_{n=0}^{\infty} n(-x^2)^n + \sum_{n=0}^{\infty} (-x^2)^n \right)$$

$$= \frac{-c_0 x^2}{(1+x^2)^2} + \frac{c_0}{(1+x^2)}$$

$$y_2(x) = \frac{c_1 x}{3} \sum_{n=0}^{\infty} (-1)^n (2n+3) x^{2n} = \frac{c_1 x}{3} \left(2\sum_{n=0}^{\infty} n(-x^2)^n + 3\sum_{n=0}^{\infty} (-x^2)^n \right)$$

$$= \frac{-2c_1 x^3}{3(1+x^2)^2} + \frac{c_1 x}{(1+x^2)}$$

So,

$$y_1(x) = c_0 \left(\frac{1}{(1+x^2)^2} \right) \quad y_2(x) = c_1 \left(\frac{3x + x^3}{3(1+x^2)^2} \right)$$

and our general solution is:

$$\boxed{y(x) = \frac{c_0}{(1+x^2)^2} + \frac{c_1}{3} \left(\frac{3x+x^3}{(1+x^2)^2} \right)}$$

where this is all assuming $|x|<1$.

Fourier Series (10 points)

The values of the periodic function $f(t)$ in one full period are given. Find the function's Fourier series.

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \\ 0 & t = \{-2, 0\} \end{cases}$$

The period here is $4 = 2L$, so the Fourier coefficients are:

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)\,dt$$

$$= \frac{1}{2}\int_{-2}^{0}(-1)\,dt + \frac{1}{2}\int_{0}^{2} 1\,dt = -1 + 1 = 0.$$

In fact, $f(t)$ is odd, so each a_n term is 0. As for the b_n terms, we get:

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t)\sin\left(\frac{n\pi t}{2}\right)dt$$

$$= \int_{0}^{2} f(t)\sin\left(\frac{n\pi t}{2}\right)dt$$ noting $f(t)$ is odd, so $f(t)\sin\left(\frac{n\pi t}{2}\right)$ is even.

$$= \int_{0}^{2}\sin\left(\frac{n\pi t}{2}\right)dt = -\frac{2}{n\pi}\cos\left(\frac{n\pi t}{2}\right)\Bigg|_{0}^{2}$$

$$= -\frac{2}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & \text{for } n \text{ } \cancel{\text{odd}} \text{ even} \\ \frac{4}{n\pi} & \text{for } n \text{ } \cancel{\text{even}} \text{ odd} \end{cases}$$

Continued...

So, our Fourier series is:

$$f(t) \sim \frac{4}{\pi}\left(\sin\left(\frac{\pi t}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{2}\right) + \cdots \right)$$

$$= \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin\left(\frac{n\pi t}{2}\right)}{n} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{(2n-1)\pi t}{2}\right)}{(2n-1)}$$

Now, if we note that at $t = 1$ $f(t)$ equals its Fourier series, we get:

$$f(1) = 1 = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \cdots \right)$$

$$= \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \right)$$

$$\Rightarrow \pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \right)$$

which is the famous Leibniz formula for π.