# Math 2280 - Final Exam

University of Utah

Spring 2009

**Name:** ________________________________________

**Laplace Transforms You May Need**

*Definition*

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt.$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

**Eigenvalue Rules for Critical Points**

$\lambda_1 < \lambda_2 < 0$ Stable improper node

$\lambda_1 = \lambda_2 < 0$ Stable node or spiral point

$\lambda_1 < 0 < \lambda_2$ Unstable saddle point

$\lambda_1 = \lambda_2 > 0$ Unstable node or spiral point

$\lambda_1 > \lambda_2 > 0$ Unstable improper node

$\lambda_2, \lambda_2 = a \pm bi$, $(a < 0)$ Stable spiral point

$\lambda_1, \lambda_2 = a \pm bi$, $(a > 0)$ Unstable spiral point

$\lambda_1, \lambda_2 = \pm bi$ Stable or unstable, center or spiral point

**Fourier Series Definition**

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

**Basic Definitions** (5 points)

Circle or state the correct answer for the questions about the following differential equation:

$$x^2y'' - \sin{(x)}y' + y^3 = e^{2x}$$

(1 point) The differential equation is: Linear Nonlinear

(1 points) The order of the differential equation is:

For the differential equation:

$$(x^4 - x)y^{(3)} + 2xe^x y' - 3y = \sqrt{x - \cos{(x)}}$$

(1 point) The differential equation is: Linear Nonlinear

(1 point) The order of the differential equation is:

(1 point) The corresponding homogeneous equation is:

**Separable Equations** (5 points)

Find the general solution to the differential equation:

$$\frac{dy}{dx} = 3\sqrt{xy}$$

**Linear First-Order Equations** (5 points)

Find the particular solution to the differential equation below with the given value:

$$xy' + 3y = 2x^5;$$

$$y(2) = 1.$$

Continued...

**Higher Order Linear Differential Equations** (5 points)

Find the general solution to the linear differential equation:

$$y'' - 3y' + 2y = 0.$$

**Nonhomogeneous Linear Differential Equations** (10 points)

Find the general solution to the differential equation:

$$y^{(3)} + 4y' = 3x - 1.$$

Continued...

**Systems of Differential Equations** (10 points)

Find the general solution to the system of differential equations:

$$\begin{array}{rcrcrcr} x_1' & = & 5x_1 & + & x_2 & + & 3x_3 \\ x_2' & = & x_1 & + & 7x_2 & + & x_3 \\ x_3' & = & 3x_1 & + & x_2 & + & 5x_3 \end{array}$$

*Hint* : $\lambda = 2$ is an eigenvalue of the coefficient matrix, and all eigenvalues are real.

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**Systems of Differential Equations with Repeated Eigenvalues** (5 points)

Find the general solution to the system of differential equations:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix} \mathbf{x}.$$

Continued...

**Laplace Transforms** (5 points)

Using the definition of the Laplace transform calculate the Laplace transform of the function:

$$f(t) = e^{3t+1}.$$

**Laplace Transforms and Differential Equations** (8 points)

Find the particular solution to the differential equation:

$$x'' + 4x = \delta(t) + \delta(t - \pi);$$

$$x(0) = x'(0) = 0.$$

Continued...

**Nonlinear Systems** (7 points)

Determine the location of the critical point $(x_0, y_0)$ for the system given below, and classify the critical point as to its type and stability.

$$\frac{dx}{dt} = x + y - 7,$$

$$\frac{dy}{dt} = 3x - y - 5.$$

Continued...

**More Nonlinear Systems** (10 points)

For the nonlinear system below, determine all critical points, and classify each according to its type and stability.

$$\frac{dx}{dt} = 3x - x^2 + \frac{1}{2}xy,$$

$$\frac{dy}{dt} = \frac{1}{5}xy - y.$$

Continued...

Continued...

**Ordinary, Regular, and Irregular Points** (5 points)

Determine if the point $x = 0$ in the following second order differential equation is an ordinary point, a regular singular point, or an irregular singular point.

$$x^3y'' + 6\sin(x)y' + 6xy = 0.$$

**Power Series Solutions** (10 points)

Find a general solution in powers of $x$ to the differential equation:

$$(x^2 + 1)y'' + 6xy' + 4y = 0.$$

Continued...

Continued...

**Fourier Series** (10 points)

The values of the periodic function $f(t)$ in one full period are given. Find the function's Fourier series.

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \\ 0 & t = \{-2, 0\} \end{cases}$$

*Extra Credit* (2 points) - Use this solution and what you know about Fourier series to deduce the famous Leibniz formula for $\pi$.

Continued...