Math 2280 - Exam ~~2~~ 3

University of Utah

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Name: Solutions

Defective Eigenvalues - Solve the system of ODEs:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 18 & 7 & 4 \\ -27 & -9 & -5 \end{pmatrix} \mathbf{x}.$$

(10 points).

The characteristic equation is:

$$(1-\lambda)\left[(7-\lambda)(-5-\lambda)+36\right]=0$$

$$\Rightarrow -(1-\lambda)^3=0$$

So, $\lambda=1$ is the only eigenvalue calculating eigenvectors:

$$\begin{pmatrix} 1 & 0 & 0 \\ 18 & 7 & 4 \\ -27 & -9 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{matrix} a=a \\ 18a+7b+4c=b \\ -27a-9b-5c=c \end{matrix}$$

If $b=0$ we get $\begin{pmatrix} 2 \\ 0 \\ -9 \end{pmatrix}$ as an ~~eigen value.~~ eigenvector.

If $c=0$ we get $\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ as an eigenvector.

To find a third solution we need to build a chain

Continued...

$$(A-\lambda I)^2 = (A-I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{pmatrix}^2$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Any vector that's not an eigenvector will work.

Take $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -9 \end{pmatrix}$$

So, our solutions are:

$$\vec{x}_1(t) = \begin{pmatrix} 2 \\ 0 \\ -9 \end{pmatrix} e^t$$

$$\vec{x}_2(t) = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} e^t$$

$$\vec{x}_3(t) = \left[\begin{pmatrix} 0 \\ 6 \\ -9 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] e^t$$

$$\boxed{\vec{x}(t) = \left[c_1\begin{pmatrix} 2 \\ 0 \\ -9 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + c_3\left(\begin{pmatrix} 0 \\ 6 \\ -9 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)\right] e^t}$$

Matrix Exponentials - Calculate e^A for the matrix:

$$A = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix}.$$

(5 points).

$$A^2 = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix}\begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } n \geq 3.$$

$$e^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix} t^2$$

$$= \boxed{\begin{pmatrix} 1+3t & 0 & -3t \\ 5t+18t^2 & 1 & 7t-18t^2 \\ 3t & 0 & 1-3t \end{pmatrix}}$$

Undetermined Coefficients - Apply the method of undetermined coefficients to find a particular solution for the system of ODEs:

$$x' = x - 5y + 2\sin t,$$

$$y' = x - y - 3\cos t.$$

(5 points).

We guess:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\vec{X}_p = \vec{a}\sin(t) + \vec{b}\cos(t)$$

$$\vec{X}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\vec{X} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin(t) + \begin{pmatrix} 0 \\ -3 \end{pmatrix}\cos(t)$$

$$X_p' = \vec{a}\cos t - \vec{b}\sin(t)$$

$$\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\vec{X}_p = \begin{pmatrix} a_1 - 5a_2 \\ a_1 - a_2 \end{pmatrix}\sin(t) + \begin{pmatrix} b_1 - 5b_2 \\ b_1 - b_2 \end{pmatrix}\cos(t)$$

$$\Rightarrow \quad -b_1 = a_1 - 5a_2 + 2 \qquad a_1 = b_1 - 5b_2$$

$$-b_2 = a_1 - a_2 \qquad a_2 = b_1 - b_2 - 3$$

$$\Rightarrow \quad a_1 = -a_1 + 5a_2 - 2 + 5a_1 - 5a_2 = 4a_1 - 2 \qquad a_1 = \frac{2}{3}$$

$$a_2 = -a_1 + 5a_2 - 2 + a_1 - a_2 - 3 = 4a_2 - 5 \qquad a_2 = \cancel{4}\ 5/3$$

$$b_1 = -a_1 + 5a_2 - 2 = -\frac{2}{3} + \frac{25}{3} - \frac{6}{3} = \frac{17}{3}$$

$$b_2 = a_2 - a_1 = \frac{3}{3} = 1$$

$$\Rightarrow \quad \boxed{\vec{X}_p = \frac{1}{3}\begin{pmatrix} 2 \\ 5 \end{pmatrix}\sin(t) + \frac{1}{3}\begin{pmatrix} 17 \\ 3 \end{pmatrix}\cos(t)}$$

Laplace Transforms - Calculate the Laplace transform of the function:

$$f(t) = t^2$$

directly from the definition of the Laplace transform. (5 points).

$$\mathcal{L}(f(t)) = \int_0^\infty t^2 e^{-st}\,dt$$

$$= -\frac{t^2 e^{-st}}{s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty t e^{-st}\,dt$$

$$= 0 + \frac{2}{s}\int_0^\infty t e^{-st}\,dt$$

$$= -\frac{2}{s^2} t e^{-st}\Big|_0^\infty + \frac{2}{s^2}\int_0^\infty e^{-st}\,dt$$

$$= -\frac{2}{s^3} e^{-st}\Big|_0^\infty$$

$$= \boxed{\frac{2}{s^3}} \qquad s > 0$$

Solving ODEs with Laplace Transforms - Use Laplace transform methods to solve the initial value problem:

$$x'' \overset{-}{\cancel{+}} 6x' + 8x = 2;$$

$$x(0) = x'(0) = 0.$$

(10 points)

$$\mathcal{L}(x'') = s^2 X(s) - s x(0) - x'(0) = s^2 X(s)$$

$$\mathcal{L}(x') = s X(s) - x(0) = s X(s)$$

$$\mathcal{L}(x) = X(s) \qquad \mathcal{L}(2) = \frac{2}{s}$$

$$\Rightarrow \cancel{s^2 + s} \quad (s^2 - 6s + 8) X(s) = \frac{2}{s}$$

$$X(s) = \frac{2}{s(s-4)(s-2)} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s-2}$$

$$= \frac{A(s-4)(s-2) + Bs(s-2) + Cs(s-4)}{s(s-4)(s-2)}$$

$$\Rightarrow (A + B + C)s^2 + (-6A - 2B - 4C)s + (8A) = 2$$

$$\Rightarrow A = \frac{1}{4} \qquad \begin{aligned} B + C &= -\frac{1}{4} \\ -2B - 4C &= \frac{3}{2} \end{aligned} \qquad \Rightarrow -2C = 1$$

$$\Rightarrow C = -1/2$$

$$B = \frac{1}{4}$$

Continued...

$$\Rightarrow \quad X(s) = \frac{\frac{1}{4}}{s} + \frac{\frac{1}{4}}{s-4} - \frac{\frac{1}{2}}{s-2}$$

$$\Rightarrow \boxed{x(t) = \frac{1}{4} + \frac{1}{4}e^{4t} - \frac{1}{2}e^{2t}}$$

Convolutions and Products - Using the definition of convolution calculate the convolution product:

$$f(t) * g(t)$$

where $f(t) = t^2$ and $g(t) = t$. (7 points)

$$t^2 * t = \int_0^t \tau^2 (t - \tau) d\tau$$

$$= t \int_0^t \tau^2 d\tau - \int_0^t \tau^3 d\tau$$

$$= \frac{t^4}{3} - \frac{t^4}{4} = \boxed{\frac{t^4}{12}}$$

Calculate the Laplace transform $\mathcal{L}(f(t) * g(t))$. (3 points)

$$\mathcal{L}(f(t) * g(t)) = \mathcal{L}(f(t)) \mathcal{L}(g(t))$$

$$= \frac{2}{s^3} \cdot \frac{1}{s^2} = \boxed{\frac{2}{s^5}}$$

Delta Functions - Solve the initial value problem:

$$x'' + 2x' + x = \delta(t) - \delta(t-2);$$

$$x(0) = x'(0) = 2.$$

(10 points).

Taking Laplace transforms of both sides:

$\mathcal{L}(x) = X(s) \quad \mathcal{L}(x') = sX(s) - x(0)$

$\mathcal{L}(x'') = s^2X(s) - sx(0) - x'(0)$

$\mathcal{L}(\delta(t-a)) = e^{-as}$

$\Rightarrow \quad s^2X(s) - 2s - 2 + 2sX(s) - 4 + X(s) = 1 - e^{-2s}$

$\Rightarrow \quad (s^2+2s+1)X(s) = 7 + 2s - e^{-as} = 2(s+1) + 5 - e^{-2s}$

$\Rightarrow \quad X(s) = \dfrac{2}{s+1} + \dfrac{5}{(s+1)^2} - \dfrac{e^{-2s}}{(s+1)^2}$

$\mathcal{L}^{-1}\left(\dfrac{1}{s+1}\right) = e^{-t} \quad \mathcal{L}^{-1}\left(\dfrac{1}{(s+1)^2}\right) = te^{-t}$

$\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = u(t-a)f(t-a) \Rightarrow \mathcal{L}^{-1}\left(\dfrac{e^{-2s}}{(s+1)^2}\right) = u(t-2)(t-2)e^{-(t-2)}$

$\Rightarrow \boxed{x(t) = 2e^{-t} + 5te^{-t} - u(t-2)(t-2)e^{-(t-2)}}$