Math 2280 - Exam 2

University of Utah

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(Solutions)

Existance and Uniqueness - State whether we're certain (based on our existance and uniqueness theorem for linear differential equations) a unique solution exists for the following differential equations on the given interval. Explain why. (5 points)

1. (1 point)

$$y'' - x(y')^2 + e^x y = 2x^2 - 5;$$

for all $x \in \mathbb{R}$.

Our existance and uniqueness theorems tell us nothing, as this is a non-linear equation. So, we don't know a unique solution exists.

2. (2 points)

$$xy'' - e^x y' + \cos xy = 25x^3;$$

for all $x > 1$.

Making the equation monic we get:

$$y'' - \frac{e^x}{x} + \frac{\cos x}{x} y = 25x^2$$

where $-\frac{e^x}{x}$, $\frac{\cos x}{x}$, and $25x^2$ are all continuous for $x > 1$. So, we know there exists a unique solution on the interval $x > 1$.

3. (2 points)

$$xy'' - e^x y' + \cos x y = 25x^3;$$

for all $x < 1$.

Converting to a monic linear ODE we get:

$$y'' - \frac{e^x}{x} + \frac{\cos x}{x} y = 25x^2$$

where $-\frac{e^x}{x}$, $\frac{\cos x}{x}$ are not continuous (or even defined) for $x=0$. So, we don't know for sure based on our theorem if this ODE has a unique solution for all $x < 1$.

Linear Differential Equations with Constant Coefficients (10 points)

1. Find the general solution to the following homogeneous differential equation: (3 points)

$$y'' - y' - 6y = 0$$

The characteristic polynomial for this linear homogeneous ODE is:

$$x^2 - x - 6 = (x-3)(x+2)$$

which has roots $r = 3, -2$.

So, the general solution is:

$$\boxed{y(x) = c_1 e^{3x} + c_2 e^{-2x}}$$

2. Use this result to calculate the general solution to the nonhomogeneous differential equation: (4 points)

$$y'' - y' - 6y = 2x + e^{-2x}$$

We can use undetermined coefficients to figure out our particular solution will be of the form:

$$y_p = Ax + B + Cxe^{-2x}$$

where we need Cxe^{-2x} and <u>not</u> Ce^{-2x} as e^{-2x} is a homogeneous solution.

So,

$$y_p = Ax + B + Cxe^{-2x}$$
$$y_p' = A - 2Cxe^{-2x} + Ce^{-2x}$$
$$y_p'' = 4Cxe^{-2x} - 4Ce^{-2x}$$

So,

$$y_p'' - y_p' - 6y_p = -5Ce^{-2x} - A - 6Ax - 6B$$
$$= 2x + e^{-2x}$$

$$\Rightarrow \begin{aligned} -5C &= 1 \\ -6A &= 2 \\ -A - 6B &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} C &= -1/5 \\ A &= -1/3 \\ B &= 1/18 \end{aligned}$$

and our general solution is:

$$\boxed{y = -\frac{1}{3}x + \frac{1}{18} - \frac{1}{5}xe^{-2x} + C_1 e^{3x} + C_2 e^{-2x}}$$

3. Find the unique solution to the following initial value problem: (3 points)

$$y'' - y' - 6y = 2x + e^{-2x}$$

$$y(0) = 2,\ y'(0) = \frac{7}{15}$$

$y(0) = \frac{1}{18} + c_1 + c_2 = 2$

$y'(0) = -\frac{1}{3} - \frac{1}{5} + 3c_1 - 2c_2 = 7/15$

$\Rightarrow \quad c_1 + c_2 = 35/18$

$3c_1 - 2c_2 = 1$

$\Rightarrow \quad 2c_1 + 2c_2 = 35/9$

$3c_1 - 2c_2 = 1$

$\Rightarrow 5c_1 = 44/9$

$\Rightarrow c_1 = 44/45$

$c_2 = \frac{35}{18} - \frac{44}{45} = \frac{175-88}{90} = \frac{87}{90} = \frac{29}{30}$

$$\Rightarrow \boxed{y = -\frac{1}{3}x + \frac{1}{18} - \frac{1}{5}xe^{-2x} + \frac{44}{45}e^{3x} + \frac{29}{30}e^{-2x}}$$

Wronskians - Calculate the Wronskian for the following sets of functions, and determine if the functions are linearly independent. If the functions are not linearly independent, demonstrate a non-trivial linear combination that equals 0. (5 points)

1. (2 points)

$$y_1 = e^{3x}$$
$$y_2 = xe^{3x}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix}$$

$$= 3xe^{6x} + e^{6x} - 3xe^{6x} = \boxed{e^{6x}}$$

$W(0) = 1 \neq 0$.

So, y_1, y_2 are $\boxed{\text{linearly independent.}}$

2. (3 points)

$$y_1 = \sin 2x$$

$$y_2 = \sin x \cos x$$

$$W(y_1, y_2) = \begin{vmatrix} \sin(2x) & \sin x \cos x \\ 2\cos(2x) & -\sin^2 x + \cos^2 x \end{vmatrix}$$

$$= (\cos^2 x - \sin^2 x)\sin(2x) - 2\sin(x)\cos(x)\cos(2x)$$

where if we plug in $x = 0$ we get $W(0) = 0$.

In fact, given $\cos^2 x - \sin^2 x = \cos(2x)$ and
$\sin(2x) = 2\sin(x)\cos(x)$

we get $W(x) = 0$. So, $\boxed{\text{linearly dependent.}}$

Given $\sin(2x) = 2\sin x \cos x$ we have:

$$\boxed{y_1 - 2y_2 = 0}$$

Converting to First-Order Systems - Convert the following system of equations into an equivlent system of first-order equations: (5 points)

$$x^{(3)} = x'' - 2x' + 5y' + 2x + 1$$

$$y'' = x' + 5x - 14y'$$

We define:

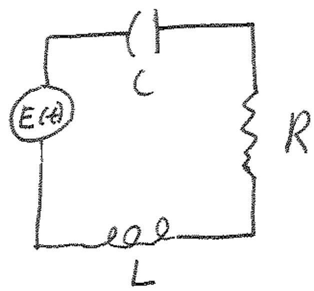
$x_1 = x'$ $\quad y_1 = y'$

$x_2 = x_1'$

Then, we get the system:

$$\boxed{\begin{aligned} x' &= x_1 \\ x_1' &= x_2 \\ x_2' &= 2x - 2x_1 + x_2 + 5y_1 + 1 \\ y' &= y_1 \\ y_1' &= 5x + x_1 - 14y_1 \end{aligned}}$$

Circuits Calculate the steady periodic current for the circuit pictured below: (10 points)



with the following parameters: $R = 200\Omega$, $L = 5H$, $C = .001F$, and $E(t) = 100\sin(10t)V$.

Kirchoff's Second Law gives us:

$$E(t) = -L\frac{dI}{dt} - RI - \frac{1}{C}Q$$

if we note $\frac{dQ}{dt} = I$ then we get:

$$E'(t) = -L\frac{d^2I}{dt^2} - R\frac{dI}{dt} - \frac{1}{C}I$$

$$\Rightarrow \quad 1000\cos(10t) = -5\frac{d^2I}{dt^2} - 200\frac{dI}{dt} - 1000I$$

$$\Rightarrow \quad \frac{d^2I}{dt^2} + 40\frac{dI}{dt} + 200I = -200\cos(10t)$$

The steady periodic current is the particular solution, and so we say:

Continued...

$$I_{sp} = A\cos(10t) + B\sin(10t)$$

$$I'_{sp} = -10A\sin(10t) + 10B\cos(10t)$$

$$I''_{sp} = -100A\cos(10t) - 100B\sin(10t)$$

Plugging this into our ODE we get:

$$-100A\cos(10t) - 100B\sin(10t) - 400A\sin(10t) + 400B\cos(10t)$$
$$+200A\cos(10t) + 200B\sin(10t) = -200\cos(10t)$$

$$\Rightarrow -A\cos(10t) - B\sin(10t) - 4A\sin(10t) + 4B\cos(10t)$$
$$+2A\cos(10t) + 2B\sin(10t) = -2\cos(10t)$$

$$\Rightarrow \begin{array}{l} A + 4B = -2 \\ -4A + B = 0 \end{array} \quad \Rightarrow 17A = -2 \Rightarrow A = -\frac{2}{17}$$

$$4B = -\frac{32}{17} \Rightarrow B = -\frac{8}{17}$$

$$\Rightarrow \boxed{I_{sp}(t) = -\frac{2}{17}\cos(10t) - \frac{8}{17}\sin(10t)}$$

First-Order Systems Solve the system of first-order differential equations given below: (10 points)

$$x_1' = 3x_1 + x_2 + x_3$$

$$x_2' = -5x_1 - 3x_2 - x_3$$

$$x_3' = 5x_1 + 5x_2 + 3x_3$$

The matrix form of this system is:

$$\vec{x}' = \begin{pmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{pmatrix} \vec{x}$$

So, the eigenvalues will be:

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ -5 & -3-\lambda & -1 \\ 5 & 5 & 3-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 + 4\lambda - 12$$
$$= (3-\lambda)(\lambda+2)(\lambda-2)$$

So, the eigenvalues are $\lambda = 3, 2, -2$ with eigenvectors

$\lambda = 3$

$$3a + b + c = 3a$$
$$-5a - 3b - c = 3b$$
$$5a + 5b + 3c = 3c$$
$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

works

$\lambda = 2$

$$3a + b + c = 2a$$
$$-5a - 3b - c = 2b$$
$$5a + 5b + 3c = 2c$$
$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

works

$\lambda = -2$

$$3a + b + c = -2a$$
$$-5a - 3b - c = -2b$$
$$5a + 5b + 3c = -2c$$
$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

works.

So,

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-2t}}$$

Is the general solution

Method of Undetermined Coefficients Find the form of the particular solution (but don't calculate the constants) for the nonhomogeneous linear differential equation given below using the method of undetermined coefficients: (5 points)

$$y^{(4)} - 2y'' + 3y' - 10y = x^3 e^{-x} \cos 4x$$

$$y_p = (Ax^3 + Bx^2 + Cx + D)e^{-x}\cos(4x)$$
$$+ (Ex^3 + Fx^2 + Gx + H)e^{-x}\sin(4x)$$