Math 2280 - Exam 1

University of Utah

Spring 2009

Name: Solutions

1. **Basics** (5 points)

For the differential equation:

$$e^{2x-4}(y'')y - \sin xy' = 7$$

What is the order of this differential equation? (2 point) 2nd

Is this differential equation linear? (1 point) No

For the differential equation:

$$x^2y' - xy = e^x$$

What is the order of the differential equation? (1 point) 1st

Is the differential equation linear? (1 point) Yes

2. **Separable Differential Equations**

Find solutions for the given initial value problems:

(a) (5 points)

$$\frac{dy}{dx} = 6e^{2x-y}$$

$$y(0) = 0$$

$\Rightarrow e^{y}dy = 6e^{2x}dx$

$= \quad e^{y} = 3e^{2x} + c$

$\Rightarrow \quad y = \ln(3e^{2x} + c)$

$y(0) = \ln(3+c) = 0 \Rightarrow 3+c=1 \Rightarrow c=-2$

$\Rightarrow \quad \boxed{y = \ln(3e^{2x}-2)}$

(b) (5 points)

$$\frac{dy}{dx} = 3x^2(y^2+1)$$

$$y(0) = 1$$

Note - $\int \frac{dx}{1+x^2} = \arctan x + C$

$$\Rightarrow \int \frac{dy}{y^2+1} = \int 3x^2\,dx$$

$$\Rightarrow \tan^{-1}(y) = x^3 + C$$

$$\Rightarrow y = \tan(x^3 + C)$$

$$y(0) = \tan(C) = 1 \Rightarrow C = \pi/4$$

$$\Rightarrow \boxed{y = \tan(x^3 + \pi/4)}$$

3. **First-Order Linear Differential Equations** (10 points)

Solve the given first-order linear differential equations.

(a) (5 points) Find the general solution.

$$y' - 2xy = e^{x^2}$$

$p(x) = -2x$

$e^{\int p(x)\,dx} = e^{-x^2}$

$\Rightarrow \quad e^{-x^2} y' - 2x e^{-x^2} y = (e^{-x^2})(e^{x^2})$

$\Rightarrow \quad \frac{d}{dx}\left(e^{-x^2} y\right) = 1$

$\Rightarrow \quad e^{-x^2} y = x + C$

$\Rightarrow \quad \boxed{y = x e^{x^2} + C e^{x^2}}$

(b) (5 points) Find the unique solution for the given initial condition.

$$y' = 1 + x + y + xy$$

$$y(0) = 0$$

Rewriting we get:

$$y' - (1+x)y = 1+x$$

$$p(x) = -(1+x), \quad e^{\int p(x)\,dx} = e^{-(x+\frac{x^2}{2})}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-(x+\frac{x^2}{2})}y\right) = (1+x)e^{-(x+x^2/2)}$$

$$\Rightarrow e^{-(x+\frac{x^2}{2})}y = \int (1+x)e^{-(x+x^2/2)} = -e^{-(x+x^2/2)} + C$$

$$\Rightarrow y = Ce^{x+x^2/2} - 1 \qquad y(0) = C - 1 = 0$$

$$\Rightarrow C = 1$$

$$\Rightarrow \boxed{y = e^{x+x^2/2} - 1}$$

Or: $\frac{dy}{dx} = (1+x)(1+y) \Rightarrow \frac{dy}{1+y} = (1+x)dx$

$$\Rightarrow \ln|1+y| = x + \frac{x^2}{2} + C \Rightarrow 1+y = Ce^{x+x^2/2}$$

$$\Rightarrow y = Ce^{x+x^2/2} - 1 \qquad y(0) = C - 1 = 0 \Rightarrow C = 1$$

$$\boxed{y = e^{x+x^2/2} - 1}$$

Either way works. Math is, yet again, consistent.

4. **Existence and Uniqueness** (5 points)

Does the differential equation:

$$\frac{dy}{dx} = y^2 e^{xy} + \sin x - x^3 + 2$$

Have a unique solution for any initial condition $y(a) = b$, where a and b are arbitrary real numbers? Justify your answer.

$f(x,y) = y^2 e^{xy} + \sin x - x^3 + 2$

$f_y(x,y) = xy^2 e^{xy} + 2y e^{xy}$

Both are continuous for all (x,y), and so for any initial condition $y(a) = b$ there will be a unique solution.

5. **Population Models** (5 points)

The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

$$\frac{dP}{dt} = k\sqrt{P} \qquad P(0) = 100$$

$$\frac{dP}{dt}(0) = k\sqrt{100} = 10k = 20 \Rightarrow k = 2$$

So,

$$\frac{dP}{dt} = 2\sqrt{P} \Rightarrow \frac{dP}{\sqrt{P}} = 2dt$$

$$\Rightarrow 2\sqrt{P} = 2t + C$$

$$\Rightarrow P = (t + c)^2$$

$$P(0) = c^2 = 100 \Rightarrow c = 10$$

$$P(t) = (t+10)^2$$

So, at $t = 12$

$$P(12) = (12+10)^2 = 22^2 =$$

$$\begin{array}{r} 22 \\ \times 22 \\ \hline 44 \\ 440 \\ \hline \boxed{484} \end{array}$$

6. **Autonomous Differential Equations** (5 points)

For the autonomous differential equation:

$$\frac{dx}{dt} = x^2 - 5x + 4$$

Find all critical points, draw the corresponding phase diagram, and indicate whether the critical points are stable, unstable, or semi-stable.

$x^2 - 5x + 4 = (x-4)(x-1)$

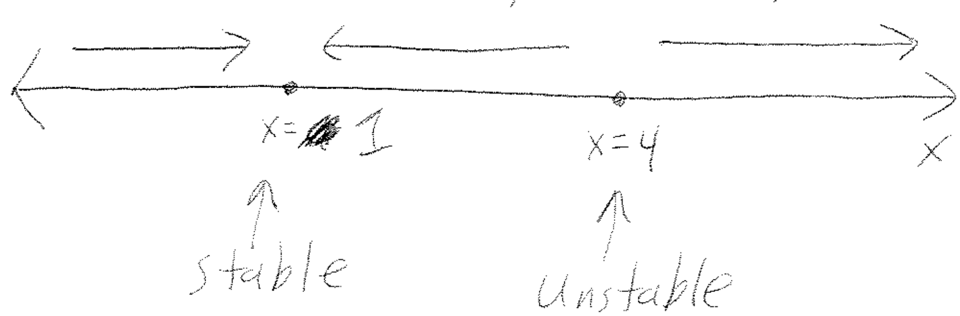
So, critical points are $x = 1$, $x = 4$

$f(x) = x^2 - 5x + 4$ $\quad f(0) = 4 \quad +$

$f(2) = -2 \quad -$

$f(5) = 4 \quad +$

So, we have the phase diagram.



7. **Euler's Method**

Use Euler's method to fill in the rest of the table for the differential equation below with the given initial conditions and step size.

Note - The y_n in the table below represent your estimated values given by Euler's method.

$$y' = 3x^2y^2;$$

$$y(0) = 1;$$

$$h = .5.$$

n	x	y_n
0	0	1
1	.5	1
2	1	1.375
3	1.5	4.211

$y_1 = 1 + .5(3(0^2)(1^2)) = 1$

$y_2 = 1 + .5(3(.5)^2(1)^2) = 11/8 = 1.375$

$y_3 = \frac{11}{8} + \frac{1}{2}\left(3(1)^2\left(\frac{11}{8}\right)^2\right) = \frac{11}{8} + \frac{363}{128} = \frac{539}{128} \approx 4.211$