5.4.1. Find a general solution to the system of equations and graph (for problem 1) the direction field and corresponding solution curves.

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix} \vec{x}$$

The system has eigenvalues:

$$\begin{vmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{vmatrix} = (2+\lambda)(4+\lambda)+1$$
$$= \lambda^2 + 6\lambda + 9 = (\lambda+3)^2$$

So, we have eigenvalue $\lambda = -3$.

We want a vector $\vec{v}_2$ such that:

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}^2 \vec{v}_2 = 0$$

but

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \vec{v}_2 = \vec{v}_1 \neq 0.$$

Now,

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, if we take $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

we get

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and the corresponding solutions:

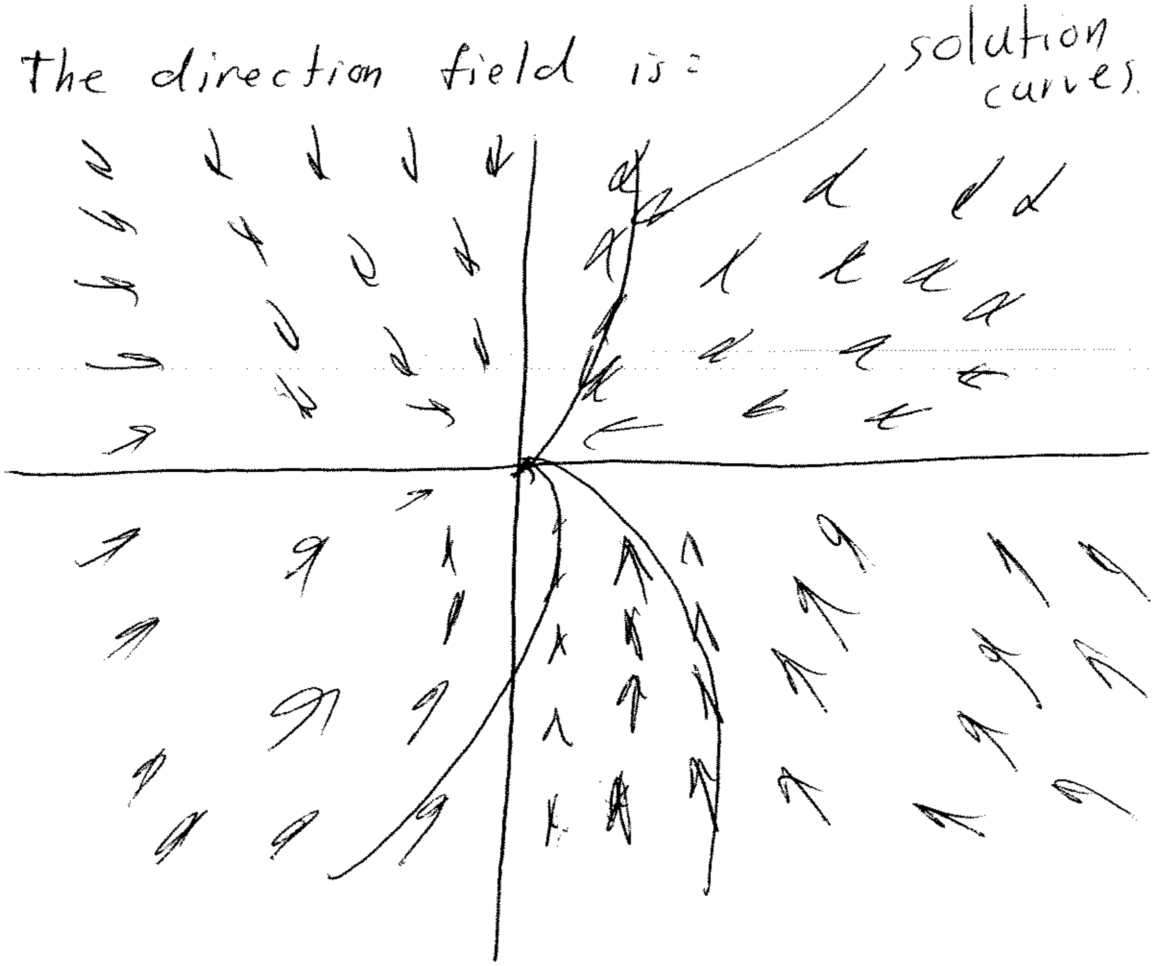
$$\vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$$

$$\vec{x}_2(t) = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{-3t}$$

So, the general solution is:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{-3t}}$$

the direction field is:



5.4.8.

$$\vec{x}' = \begin{pmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{pmatrix} \vec{x}$$

the characteristic equation is:

$$(13-\lambda)\left[(25-\lambda)(-5-\lambda)-(12)(-18)\right]$$

$$= (13-\lambda)\left(\lambda^2 - 20\lambda + 91\right)$$

$$= -(\lambda-13)^2(\lambda-7)$$ So, eigenvalues: $\lambda = 7$, $\lambda = 13$

For $\lambda = 7$ we have the eigenvector:

$$\begin{pmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 7\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 18a + 12b = 0 \\ -18a - 12b = 0 \\ 6a + 6b + 6c = 0 \end{array} \qquad \begin{array}{l} a = 2 \\ b = -3 \\ c = 1 \end{array}$$ works.

$$\vec{v} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

~~as for $\lambda = 13$ we must find two vectors satisfying:~~

$$\cancel{(A-\lambda I)^2 \vec{v}_2 = 0}$$

$$(\cancel{A - \lambda I)\vec{v}_2 = \vec{v}_1 \neq 0}.$$

Now, for $\lambda = 13$ we have eigenvectors

$$\begin{pmatrix} 12 & 12 & 0 \\ -18 & -18 & 0 \\ 6 & 6 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

both work! So, our general solution is:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{13t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{13t}}$$

5.4.15

$$\vec{x}\,' = \begin{pmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -2-\lambda & -9 & 0 \\ 1 & 4-\lambda & 0 \\ 1 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(-2-\lambda)(4-\lambda) + 9\right]$$

$$= (1-\lambda)(\lambda^2 - 2\lambda + 1)$$
$$= (1-\lambda)(\lambda - 1)^2 = -(\lambda - 1)^3$$

So, we have the eigenvalue $\lambda = 1$ which has order 3.

So, we have the eigenvector equation:

$$\begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we get two linearly independent eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for example}$$

Now, one must be "defective" and I'll bet it's the first. So, we get:

$$\begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, any vector works as long as

$$\begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \vec{v}_2 \neq 0.$$

Take $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then

$$\vec{v}_1 = \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

So, we get the general solution:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] e^{t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t}}$$

4.25

$$\vec{x}' = \begin{pmatrix} -2 & 17 & 4 \\ -1 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix} \vec{x}$$

with eigenvalues (given in problem) $\lambda = 2, 2, 2$.

So, we have the eigenvector equation:

$$(A - \lambda I) = \begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Eigenvectors that work are:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$ and that's it!

So, we take:

$$\begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & 1 \end{pmatrix} = (A - \lambda I)^2$$

Now,

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & 1 \end{pmatrix}\begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, any <u>vector</u> $\vec{v}_3$ that satisfies

$$(A - \lambda I)^2 \vec{v}_3 \neq 0 \quad \cancel{\text{and } (A-\lambda I}\text{)}$$

will work. Take $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

we then get:

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \vec{v}_1$$

and

$$\begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} = \vec{v}_2$$

So, we get solutions:

$$\vec{x}_1(t) = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} e^{2t}$$

$$\vec{x}_2(t) = \left[\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix}\right] e^{2t}$$

$$\vec{x}_3(t) = \left[\frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] e^{2t}$$

So, our general solution is:

$$\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} \right] e^{2t} + c_3 \left[\frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} t^2 + \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] e^{2t}$$

5.4.33 The characteristic equation of the coefficient matrix A of the system

$$\vec{x}' = \begin{pmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{pmatrix} \vec{x}$$

is

$$\phi(\lambda) = (\lambda^2 - 6\lambda + 25)^2 = 0$$

Therefore, A has the repeated complex conjugate pair $3 \pm 4i$ of eigenvalues. First show that the complex vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix}$$

Note: There is a typo in the book! The top entry here should be 0, not 9!

form a length 2 chain $\{\vec{v}_1, \vec{v}_2\}$ associated with the eigenvalue $\lambda = 3-4i$. Then calculate the real and imaginary parts of the complex-valued solutions $\vec{v}_1 e^{\lambda t}$ and $(\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$

to find four independent real-valued solutions of $\vec{x}' = A\vec{x}$.

First, we verify it's a chain:

$$(A - \lambda I) = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix}$$

$$(A - \lambda I)\vec{v}_2 = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} = \vec{v}_1$$

$$(A - \lambda I)\vec{v}_1 = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, it's a chain.

So, we have solutions:

$$\vec{X}_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} e^{(3-4i)t} \quad \vec{X}_2 = \left[\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} \right] e^{(3-4i)t}$$

Breaking this up into its real and complex parts we get:

$$e^{(3-4i)t} = e^{3t}\left(\cos(4t) - i\sin(4t)\right)$$

$$Re(\vec{x}_1) = \begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t}$$

$$Im(\vec{x}_1) = \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t}$$

$$Re(\vec{x}_2) = \begin{pmatrix} t\cos(4t) \\ t\sin(4t) \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t}$$

$$Im(\vec{x}_2) = \begin{pmatrix} -t\sin(4t) \\ t\cos(4t) \\ -\sin(4t) \\ \cos(4t) \end{pmatrix} e^{3t}$$

So, our final solution is:

$$\boxed{\vec{X}(t) = c_1 \begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} t\cos(4t) \\ t\sin(4t) \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t} + c_4 \begin{pmatrix} -t\sin(4t) \\ t\cos(4t) \\ -\sin(4t) \\ \cos(4t) \end{pmatrix}}$$

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Math 2280
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Assignment #9 Solutions

Section 5.5

5.5.1 Find a fundamental matrix and a solution to the system

$$\vec{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Solve for the eigenvalues of the matrix:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$$

$$\Rightarrow \quad 2-\lambda = \pm 1 \quad \Rightarrow \quad \boxed{\lambda = 1, 3}$$

Finding an eigenvector for $\lambda = 1$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$a+b=0$
$a+b=0$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works

Finding an eigenvector for $\lambda = 3$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix}$$

$-a+b=0$
$-a+b=0$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works.

So, we get two linearly independent solutions:

$$\vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$$

$$\vec{x}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

And a fundamental matrix:

$$\Phi(t) = \begin{pmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{pmatrix}$$

Now, $\Phi(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

$$\Rightarrow \quad \Phi(0)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \quad \vec{c} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix}$$

and so,

$$\vec{x}(t) = \begin{pmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{5}{2} e^{t} + \frac{1}{2} e^{3t} \\ -\frac{5}{2} e^{t} + \frac{1}{2} e^{3t} \end{pmatrix}}$$

5.5.7.

$$\vec{x}' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

First, we find the eigenvalues of the matrix:

$$\begin{vmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{vmatrix}$$

$$= (5-\lambda)[(1+\lambda)(4+\lambda)-4] - 6[-4+4(1+\lambda)]$$

$$= (5-\lambda)[\lambda^2+5\lambda] - 24\lambda$$

$$= 5\lambda^2 + 25\lambda - \lambda^3 - 5\lambda^2 - \cancel{6\lambda}\ 24\lambda$$

$$= \cancel{19\lambda - \lambda^3}\ \lambda - \lambda^3$$

$$= \lambda(1-\lambda)(1+\lambda)$$

So, we have eigenvalues $\lambda = 0, -1, 1$

For $\lambda = 0$ we have the eigenvector

$$\begin{aligned} 5a \qquad - 6c &= 0 \\ 2a - b - 2c &= 0 \\ 4a - 2b - 4c &= 0 \end{aligned} \qquad \vec{x} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \text{ works.}$$

For $\lambda = 1$ we have:

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{aligned} 4a \quad -6c &= 0 \\ 2a - 2b - 2c &= 0 \\ 4a - 2b - 5c &= 0 \end{aligned} \qquad \vec{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \text{ works.}$$

For $\lambda = -1$ we have:

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{aligned} 6a \quad -6c &= 0 \\ 2a \quad -2c &= 0 \\ 4a - 2b - 3c &= 0 \end{aligned} \qquad \vec{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \text{ works}$$

So, a fundamental matrix is:

$$\Phi(t) = \begin{pmatrix} 6 & 3e^{t} & 2e^{-t} \\ 2 & e^{t} & e^{-t} \\ 5 & 2e^{t} & 2e^{-t} \end{pmatrix}$$

To find a particular solution:

$$\Phi(0) = \begin{pmatrix} 6 & 3 & 2 \\ 2 & 1 & 1 \\ 5 & 2 & 2 \end{pmatrix}$$

using Matlab

$$\Phi(0)^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & 3 & 0 \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

and

$$x(t) = \begin{pmatrix} 6 & 3e^{t} & 2e^{-t} \\ 2 & e^{t} & e^{-t} \\ 5 & 2e^{t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} -12 + 12e^{t} + 2e^{-t} \\ -4 + 4e^{t} + e^{-t} \\ -10 + 8e^{t} + 2e^{-t} \end{pmatrix}$$

5.5.9 Compute the matrix exponential e^{At} for each system $\vec{x}' = A\vec{x}$.

$$x_1' = 5x_1 - 4x_2$$
$$x_2' = 2x_1 - x_2$$

$$\vec{A} = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$$

Finding the eigenvalues:

$$\begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda-5)(\lambda+1) + 8$$
$$= \lambda^2 - 4\lambda + 3$$
$$= (\lambda-3)(\lambda-1)$$

$\lambda = 3, 1$

Finding two eigenvectors for our eigenvalues:

For $\lambda=3$

$$\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix}$$

$2a - 4b = 0$ $\quad \vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ works.
$2a - 4b = 0$

For $\lambda = 1$

$$\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$4a - 4b = 0$ $\quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works.
$2a - 2b = 0$

So, a fundamental matrix is:

$$\Phi(t) = \begin{pmatrix} 2e^{3t} & e^{t} \\ e^{3t} & e^{t} \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\Phi^{-1}(0) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

and so

$$e^{At} = \Phi(t)\,\Phi(0)^{-1} = \begin{pmatrix} 2e^{3t} & e^{t} \\ e^{3t} & e^{t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 2e^{3t} - e^{t} & -2e^{3t} + 2e^{t} \\ e^{3t} - e^{t} & -e^{3t} + 2e^{t} \end{pmatrix}}$$

5.5.18

$$x_1' = 4x_1 + 2x_2$$
$$x_2' = 2x_1 + 4x_2$$

$$\vec{x}' = A\vec{x} \qquad A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

Finding the eigenvalues:

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 4$$
$$= \lambda^2 - 8\lambda + 12$$
$$= (\lambda - 6)(\lambda - 2)$$

So, the eigenvalues are $\lambda = 2$ and $\lambda = 6$.

Finding the eigenvectors:

$\lambda = 2$

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{matrix} 2a + 2b = 0 \\ 2a + 2b = 0 \end{matrix} \qquad \vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ works.}$$

$\lambda = 6$

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 6\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\rightarrow \begin{matrix} -2a + 2b = 0 \\ 2a - 2b = 0 \end{matrix} \qquad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

So, we have fundamental matrix:

$$\Phi(t) = \begin{pmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\Phi(0)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{2t} + \frac{1}{2}e^{6t} & -\frac{1}{2}e^{2t} + \frac{1}{2}e^{6t} \\ -\frac{1}{2}e^{2t} + \frac{1}{2}e^{6t} & \frac{1}{2}e^{2t} + \frac{1}{2}e^{6t} \end{pmatrix}$$

5.5.24 Show that the matrix A is nilpotent and then use this fact to find the matrix exponential e^{At}.

$$A = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix}\begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So,

$$e^{At} = I + At + \frac{1}{2!}A^2t^2$$

$$= \cancel{\begin{pmatrix} 1+3t & 0 & -3t \\ 5t+18t^2 & 1 & 7t-18t^2 \\ 3t & 0 & 1-3t \end{pmatrix}} = \begin{pmatrix} 1-3t & 0 & -3t \\ 5t+18t^2 & 1 & 7t-18t^2 \\ 3t & 0 & 1-3t \end{pmatrix}$$

5.6.1 Use the method of undetermined coefficients to find a particular solution. If initial conditions are given, find a particular solution that satisfies these conditions.

$$x' = x + 2y + 3$$
$$y' = 2x + y - 2$$

$$\Rightarrow \vec{x}\,' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

First, we solve the homogenous equation:

$$\vec{x}\,' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{x}$$

Finding the eigenvalues

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow (1-\lambda)^2 = 4$$
$$\Rightarrow (1-\lambda) = \pm 2$$
$$\lambda = \{-1, 3\}.$$

Finding eigenvectors

For $\lambda = -1$.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow 2a + 2b = 0 \qquad \vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ works.}$$
$$2a + 2b = 0$$

For $\lambda = 3$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{aligned} -2a + 2b &= 0 \\ 2a - 2b &= 0 \end{aligned} \qquad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

So, our homogenous general solution is:

$$\vec{x}_h(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}$$

Now, for our particular solution we note that it's a constant "polynomial", and so we "guess" our solution is too:

$$\vec{x}_p = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad \vec{x}_p{}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{aligned} c_1 + 2c_2 + 3 &= 0 \\ 2c_1 + c_2 - 2 &= 0 \end{aligned}$$

$$-3c_1 + 7 = 0 \Rightarrow \cancel{c_1 = \tfrac{3}{7}} \quad c_1 = 7/3$$

$$\Rightarrow \cancel{\tfrac{6}{7} + c_2 - 2 = 0 \Rightarrow c_2 = \tfrac{8}{7}}$$

$$14/3 + c_2 - 2 = 0 \Rightarrow c_2 = -8/3.$$

So,

$$\vec{x}_p = \frac{1}{3}\begin{pmatrix} 7 \\ -8 \end{pmatrix}$$

Thus, our solution is:

$$\boxed{\vec{x}(t) = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{3t} + \frac{1}{3}\begin{pmatrix}7\\-8\end{pmatrix}}$$

The particular solution can be expressed:

$$x(t) = 7/3 \qquad y(t) = -8/3.$$

5.6.6. $x' = 9x + y + 2e^t$
$y' = -8x - 2y + te^t$

So,

$$\vec{x}' = \begin{pmatrix}9 & 1\\-8 & -2\end{pmatrix}\vec{x} + \begin{pmatrix}2e^t\\te^t\end{pmatrix}$$

First we solve the homogenous problem

$$\vec{x}' = \begin{pmatrix}9 & 1\\-8 & -2\end{pmatrix}\vec{x}$$

Finding the eigenvalues:

$$\begin{vmatrix}9-\lambda & 1\\-8 & -2-\lambda\end{vmatrix} = (9-\lambda)(-2-\lambda) + 8$$
$$= -18 - 9\lambda + 2\lambda + \lambda^2 + 8$$
$$= \lambda^2 - 7\lambda - 10$$

$$\lambda = \frac{7 \pm \sqrt{(-7)^2 - 4(1)(-10)}}{2} = \frac{7}{2} \pm \frac{\sqrt{89}}{2}$$

We note that neither of these $\lambda = 1$, so there is no interference with our particular solution.

We "guess" that the form of our particular solution is

$$\vec{x}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} te^t + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} e^t$$

$$\vec{x}_p\,' = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} te^t + \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} e^t$$

So, we get the relations from our system:

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} te^t + \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} e^t = \begin{pmatrix} 9 & 1 \\ -8 & -2 \end{pmatrix} \left[\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} te^t + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} e^t \right] + \begin{pmatrix} 0 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^t$$

So, we end up with the 4 relations:

$$\begin{aligned} a_1 &= 9a_1 + b_1 \\ b_1 &= -8a_1 - 2b_1 + 1 \\ a_1 + a_2 &= 9a_2 + b_2 + 2 \\ b_1 + b_2 &= -8a_2 - 2b_2 \end{aligned}$$

Solving first for a_1 and b_1

$$b_1 = -8a_1$$

$$\Rightarrow b_1 = b_1 - 2b_1 + 1 \Rightarrow 2b_1 = 1 \Rightarrow b_1 = \tfrac{1}{2} \qquad a_1 = -\tfrac{1}{16}$$

and using these values to solve for a_2, b_2 we get:

$$a_2 = -\frac{91}{256} \qquad b_2 = \frac{25}{32}.$$

So,

$$\boxed{\vec{x}_p = \frac{1}{16}\begin{pmatrix} -1 \\ 8 \end{pmatrix} te^t + \frac{1}{256}\begin{pmatrix} -91 \\ 200 \end{pmatrix} e^t}$$

5.6.10. $x' = x - 2y$
$y' = 2x - y + e^t \sin(t)$

First, we investigate the eigenvalues of our homogenous equation:

$$\vec{x}' = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 4$$
$$= -1 - \lambda + \lambda + \lambda^2 + 4$$
$$= \lambda^2 + 3$$

So, we have eigenvalues $\lambda = \pm\sqrt{3}\, i$, which is not $\lambda = 1 \pm i$, so we have no interference with our "guess" and our homogenous solution.

Our system is:

$$\vec{x}' = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \sin(t).$$

So, our "guess" will be

$$\vec{x}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^t \cos(t) + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} e^t \sin(t)$$

$$\vec{x}_p' = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} e^t \cos(t) + \begin{pmatrix} -a_1 + a_2 \\ -b_1 + b_2 \end{pmatrix} e^t \sin(t).$$

Plugging this into our system of ODEs we get:

$$\begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} e^t \cos(t) + \begin{pmatrix} -a_1 + a_2 \\ -b_1 + b_2 \end{pmatrix} e^t \sin(t)$$

$$= \begin{pmatrix} a_1 - 2b_1 \\ 2a_1 - b_1 \end{pmatrix} e^t \cos(t) + \begin{pmatrix} a_2 - 2b_2 \\ 2a_2 - b_2 + 1 \end{pmatrix} e^t \sin(t)$$

So, we get the 4 equations:

$$\begin{aligned} a_1 + a_2 &= a_1 - 2b_1 \\ b_1 + b_2 &= 2a_1 - b_1 \\ -a_1 + a_2 &= a_2 - 2b_2 \\ -b_1 + b_2 &= 2a_2 - b_2 + 1 \end{aligned} \qquad \begin{aligned} a_2 &= -2b_1 \\ b_2 &= 2a_1 - 2b_1 \\ -a_1 &= -2b_2 \end{aligned}$$

$$\Rightarrow \begin{aligned} -a_1 - 2b_1 &= -2b_1 - 2(2a_1 - 2b_1) \\ -b_1 + (2a_1 - 2b_1) &= 2(-2b_1) - (2a_1 - 2b_1) + 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} 3a_1 - 4b_1 &= 0 \\ 4a_1 - 3b_1 &= 1 \end{aligned} \Rightarrow \begin{aligned} -12a_1 + 16b_1 &= 0 \\ 12a_1 - 9b_1 &= 3 \end{aligned}$$

$$\Rightarrow 7b_1 = 3 \Rightarrow b_1 = 3/7$$

$$a_1 = 12/21 = 4/7$$

$$a_2 = -6/7$$

$$b_2 = 2/7$$

So,

$$\boxed{\vec{x}_p(t) = \frac{1}{21}\begin{pmatrix} 12 \\ 9 \end{pmatrix} e^t \cos(t) + \frac{1}{7}\begin{pmatrix} -6 \\ 2 \end{pmatrix} e^t \sin(t)}$$

$$-a_1 - 2b_1 = -2b_1 - 4a_1 + 4b_1$$
$$2a_1 - 3b_1 = -4b_1 - 2a_1 + 2b_1 + 1$$

$$\Rightarrow \quad -a_1 - 2b_1 = -4a_1 + 2b_1$$
$$2a_1 - 3b_1 = -2a_1 - 2b_1 + 1$$

$$\Rightarrow \quad 3a_1 - 4b_1 = 0$$
$$4a_1 - b_1 = 1$$

$$\Rightarrow \quad 3a_1 - 4b_1 = 0$$
$$-16a_1 + 4b_1 = -4$$

$$\Rightarrow -13a_1 = -4 \qquad \Rightarrow a_1 = 4/13$$
$$b_2 = 2/13$$
$$b_1 = a_1 - \tfrac{1}{2}b_2 = 3/13$$
$$a_2 = -6/13$$

So,

$$\boxed{x_p(t) = \frac{1}{13}\begin{pmatrix} 4 \\ 3 \end{pmatrix} e^t \cos(t) + \frac{1}{13}\begin{pmatrix} -6 \\ 2 \end{pmatrix} e^t \sin(t)}$$

5.6.17 Use the method of variation of parameters to solve the initial value problem

$$\vec{x}' = A\vec{x} + \vec{f}(t), \qquad \vec{x}(a) = \vec{x}_a.$$

$$A = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}, \quad \vec{f}(t) = \begin{pmatrix} 60 \\ 90 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$e^{At} = \frac{1}{6}\begin{pmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{pmatrix}$$

$$e^{-A(s-t)} = \frac{1}{6}\begin{pmatrix} -e^{(s-t)} + 7e^{-5(s-t)} & 7e^{s-t} - 7e^{-5(s-t)} \\ -e^{(s-t)} + e^{-5(s-t)} & 7e^{s-t} - e^{-5(s-t)} \end{pmatrix}$$

$$\vec{x}(t) = \int_0^t e^{-A(s-t)} f(s)\,ds$$

$$= \int_0^t \begin{pmatrix} -e^{(s-t)} + 7e^{-5(s-t)} & 7e^{s-t} - 7e^{-5(s-t)} \\ -e^{(s-t)} + e^{-5(s-t)} & 7e^{s-t} - e^{-5(s-t)} \end{pmatrix}\begin{pmatrix} 10 \\ 15 \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} 95e^{(s-t)} - 35e^{-5(s-t)} \\ 95e^{(s-t)} - 5e^{-5(s-t)} \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} 95e^{-t}e^{s} - 35e^{5t}e^{-5s} \\ 95e^{-t}e^{s} - 5e^{5t}e^{-5s} \end{pmatrix} ds$$

$$= \begin{pmatrix} 95 + 7 \\ 95 + 1 \end{pmatrix} - \begin{pmatrix} 95e^{-t} + 7e^{5t} \\ 95e^{-t} + e^{5t} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 102 - 95e^{-t} \mp 7e^{5t} \\ 96 - 95e^{-t} \mp e^{5t} \end{pmatrix}}$$

5.6-19 $\quad A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \quad \vec{f}(t) = \begin{pmatrix} 180t \\ 90 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$e^{At} = \frac{1}{5}\begin{bmatrix} e^{-3t}+4e^{2t} & -2e^{-3t}+2e^{2t} \\ -2e^{-3t}+2e^{2t} & 4e^{-3t}+e^{2t} \end{bmatrix}$$

$$e^{-A(s-t)} = \frac{1}{5}\begin{bmatrix} e^{3(s-t)}+4e^{-2(s-t)} & -2e^{3(s-t)}+2e^{-2(s-t)} \\ -2e^{3(s-t)}+2e^{-2(s-t)} & 4e^{3(s-t)}+e^{-2(s-t)} \end{bmatrix}$$

$$\vec{x}(t) = \int_0^t \begin{pmatrix} e^{3(s-t)}+4e^{-2(s-t)} & -2e^{3(s-t)}+2e^{-2(s-t)} \\ -2e^{3(s-t)}+2e^{-2(s-t)} & 4e^{3(s-t)}+e^{-2(s-t)} \end{pmatrix}\begin{pmatrix} 36s \\ 18 \end{pmatrix}$$

$$= \cancel{\int_0^t e^{-3t}(32s-36)e^{3s} + e^{2t}(128s+36)}$$

$$= \int_0^t \begin{pmatrix} e^{-3t}(36s-36)e^{3s} + e^{2t}(\overset{144}{\cancel{128}}s+36)e^{-2s} \\ e^{-3t}(-\underset{72}{\cancel{64}}s+72)e^{3s} + e^{2t}(\underset{72}{\cancel{64}}s+18)e^{-2s} \end{pmatrix} ds$$

$$= \left.\begin{pmatrix} e^{-3t}(12se^{3s}-16e^{3s}) + e^{2t}(-72se^{-2s}-54e^{-2s}) \\ e^{-3t}(-24se^{3s}+32e^{3s}) + e^{2t}(-36se^{-2s}-27e^{-2s}) \end{pmatrix}\right|_0^t$$

$$= \begin{pmatrix} 12t-16-72t-54+16e^{-3t}+54e^{2t} \\ -24t+32-36t-27+32e^{-3t}+27e^{2t} \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} -60t-70+16e^{-3t}+54e^{2t} \\ -60t\ \underset{+5}{\cancel{-59}}+32e^{-3t}+27e^{2t} \end{pmatrix}}$$

$$= \boxed{\begin{pmatrix} -60t-70+16e^{-3t}+54e^{2t} \\ -60t+5+32e^{-3t}+27e^{2t} \end{pmatrix}}$$