5.1.1. Let

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix}$$

Find

a) $2A + 3B$

$$2\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} + 3\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 13 & -18 \\ 23 & 17 \end{pmatrix}$$

b) $3A - 2B$

$$3\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} - 2\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 19 \end{pmatrix}$$

c) AB

$$\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -9 & -11 \\ 47 & -9 \end{pmatrix}$$

d) BA

$$\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} -10 & -37 \\ 14 & -8 \end{pmatrix}$$

5.1-7 Compute the determinant of the matrices A and B and verify the results are consistent with:

$$\det(A)\det(B) = \det(AB)$$

$$\det(A) = 14 - (-12) = 26$$

$$\det(B) = 3 - (-20) = 23$$

$$\det(AB) = 81 - (-517) = 598 = 26 \times 23.$$

So, math is consistent. Phew!

5.1.15 Write the system in the form

$$\vec{x}\,'(t) = \vec{P}(t)\,\vec{x}(t) + \vec{f}(t)$$

$$x' = y + z$$
$$y' = z + x$$
$$z' = x + y$$

$$\Rightarrow \quad \vec{x}\,' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$$

with $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

5.1.21. Verify the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution to the system

$$\vec{x}' = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \vec{x}$$

$$\vec{x}_1 = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} \qquad \vec{x}_1' = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}$$

So,

$$\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} = \vec{x}_1'$$

So, it checks out.

$$\vec{x}_2 = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} \qquad \vec{x}_2' = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix} = \vec{x}_2'$$

So, it checks out too.

$$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = +e^{3t} \neq 0.$$ So, linearly independent

General Solution:

$$\vec{x} = c_1 \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$$

5.1.27

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$$

$$\vec{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{x}_1' = e^{2t}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e^{2t}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

So, it checks out.

$$\vec{x}_2 = e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \vec{x}_2' = e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So, it checks out.

$$\vec{x}_3 = e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{x}_3' = e^{-t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = e^{-t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

So, it checks out.

$$W(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} = 1 + 2 + 0 = 3 \neq 0$$

So, they are linearly independent

$$\boxed{\vec{x} = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$$

Dylan Zwick

Solution Set #8

5.2.1 Solve the system of ODEs and graph the corresponding direction field and typical solution curves.

$$x_1' = x_1 + 2x_2$$
$$x_2' = 2x_1 + x_2$$

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$
$$= (\lambda - 3)(\lambda + 1)$$

So, we have eigenvalues $\lambda = 3, -1$ with corresponding eigenvectors

$\lambda = 3$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{matrix} a + 2b = 3a \\ 2a + b = 3b \end{matrix} \Rightarrow \begin{matrix} -2a + 2b = 0 \\ 2a - 2b = 0 \end{matrix}$$

So,

$a = b$ and an eigenvector

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = -1$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} 2a + 2b = 0 \\ 2a + 2b = 0 \end{matrix} \quad \Rightarrow \quad a = -b$$

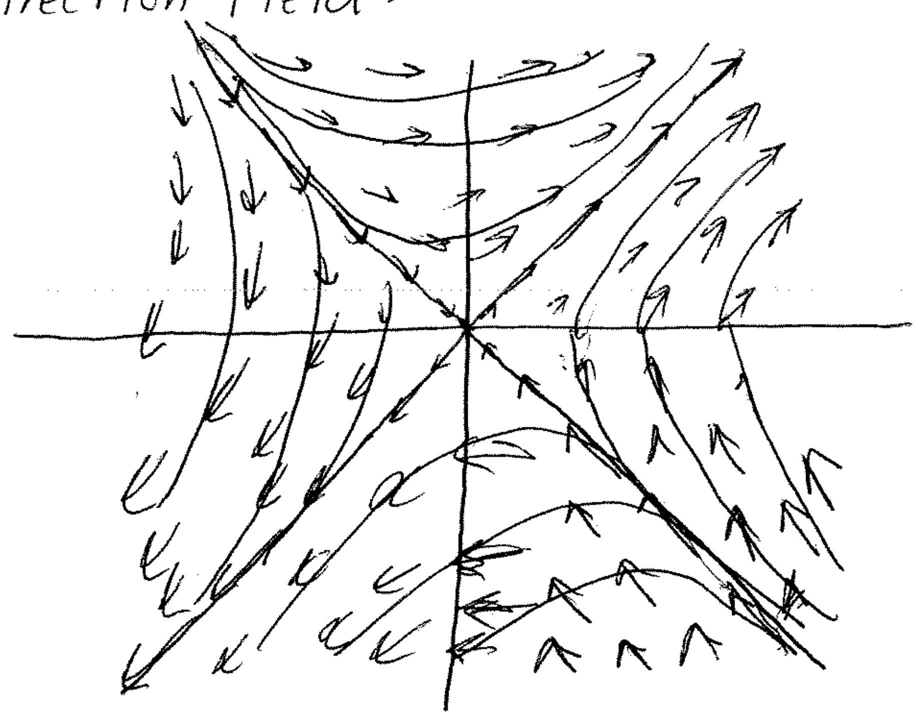
So,

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a working eigenvector.

So, we have general solution

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

Direction Field:



5.2.9

$$x_1' = 2x_1 - 5x_2 \qquad x_1(0) = 2$$
$$x_2' = 4x_1 - 2x_2 \qquad x_2(0) = 3$$

$$\vec{x}' = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} = \sout{(2-\lambda)^2 + 20 = -\lambda^2 + 4\lambda + 16}$$
$$= (2-\lambda)(-2-\lambda) + 20$$
$$= -4 + \lambda^2 + 20$$

$$\Rightarrow \lambda^2 + 16 \Rightarrow \boxed{\lambda = \pm 4i}$$

So, we have complex eigenvalues.

$$\begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix} = 4i \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix}$$

$$\Rightarrow (2a_1 - 5a_2) + i(2b_1 - 5b_2) = -4b_1 + i\,4a_1$$
$$(4a_1 - 2a_2) + i(4b_1 - 2b_2) = -4b_2 + i\,4a_2$$

So, we get the relations

$$2a_1 - 5a_2 = -4b_1$$
$$4a_1 - 2a_2 = -4b_2$$
$$2b_1 - 5b_2 = 4a_1$$
$$4b_1 - 2b_2 = 4a_2$$

One set that works is

$$a_1 = 5 \quad b_1 = 5$$
$$a_2 = 6 \quad b_2 = -2$$

~~$2b_1 - 5b_2 - 2b_1 + b_2 = -4b_2$~~

~~$\Rightarrow$~~

So, we get the general solution:

$$\vec{x}(t) = c_1\left[\begin{pmatrix}5\\6\end{pmatrix}\cos(4t) - \begin{pmatrix}5\\-2\end{pmatrix}\sin(4t)\right] + c_2\left[\begin{pmatrix}5\\-2\end{pmatrix}\cos(4t) + \begin{pmatrix}5\\6\end{pmatrix}\sin(4t)\right]$$

$x_1(0) = 2 \qquad x_2(0) = 3$

$$\Rightarrow \begin{array}{l} 5c_1 + 5c_2 = 2 \\ 6c_1 - 2c_2 = 3 \end{array} \Rightarrow \begin{array}{l} 30c_1 + 30c_2 = 12 \\ 30c_1 - 10c_2 = 15 \end{array}$$

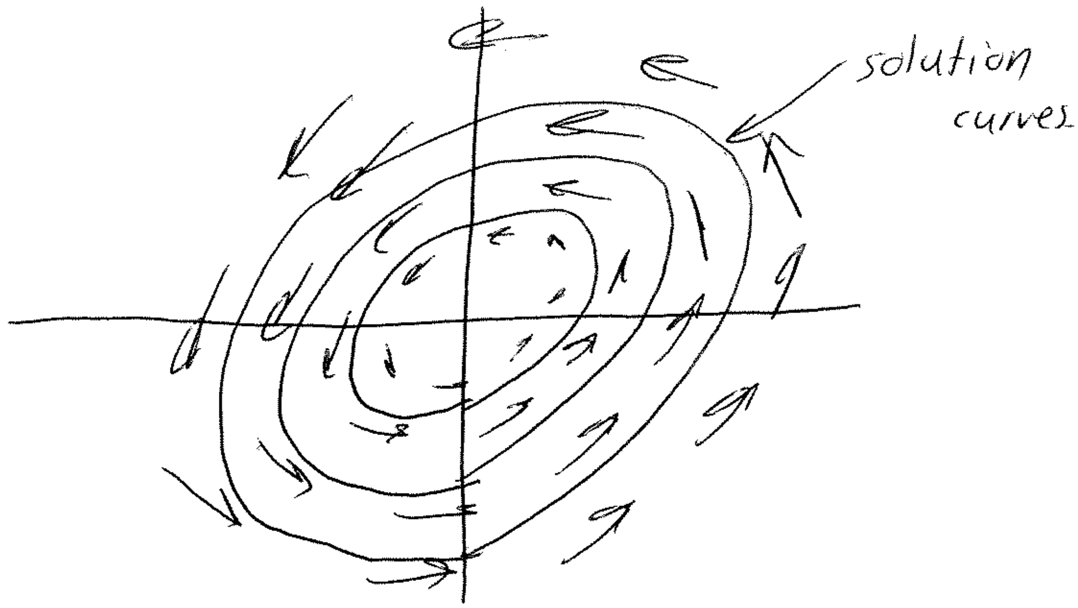
$$\Rightarrow 40c_2 = -3 \qquad c_2 = -3/40$$

$$c_1 = \frac{1}{5}\left(2 + \frac{3}{8}\right) = 19/40$$

So,

$$\vec{x}(t) = \frac{19}{40}\left[\begin{pmatrix}5\\6\end{pmatrix}\cos(4t) - \begin{pmatrix}5\\-2\end{pmatrix}\sin(4t)\right] - \frac{3}{40}\left[\begin{pmatrix}5\\-2\end{pmatrix}\cos(4t) + \begin{pmatrix}5\\6\end{pmatrix}\sin(4t)\right]$$

$$\Rightarrow \boxed{\vec{x}(t) = \begin{pmatrix} 2\cos(4t) - \frac{11}{4}\sin(4t) \\ 3\cos(4t) + \frac{1}{2}\sin(4t) \end{pmatrix}}$$



5.2.15

$x_1' = 7x_1 - 5x_2$
$x_2' = 4x_1 + 3x_2$

$$\vec{x}' = \begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix} \vec{x}$$

Solving for the eigenvalues:

$$\begin{vmatrix} 7-\lambda & -5 \\ 4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) + 20$$

$$= 21 - 10\lambda + \lambda^2 + 20$$

$$\Rightarrow \lambda^2 - 10\lambda + 41$$

$$\lambda = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(41)}}{2} = 5 \pm 4i$$

So, we want to find an eigenvector:

$$\begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix} = 5+4i \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} (7a_1 - 5a_2) + i(7b_1 - 5b_2) = (5a_1 - 4b_1) + i(4a_1 + 5b_1) \\ (4a_1 + 3a_2) + i(4b_1 + 3b_2) = (5a_2 - 4b_2) + i(4a_2 + 5b_2) \end{matrix}$$

Interestingly enough we see, again, that the set: $a_1 = 5 \quad b_1 = 5$
$a_2 = 6 \quad b_2 = -2$

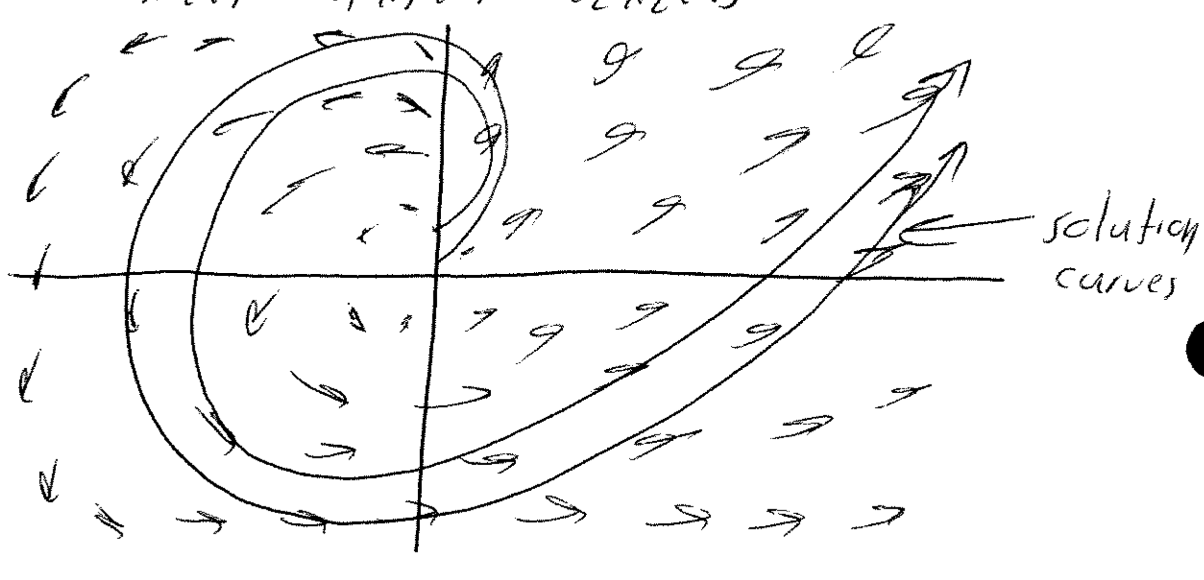
So, we have the solutions:

$$\vec{x}_1(t) = e^{5t}\left[\begin{pmatrix} 5 \\ 6 \end{pmatrix}\cos(4t) - \begin{pmatrix} 5 \\ -2 \end{pmatrix}\sin(4t)\right]$$

$$\vec{x}_2(t) = e^{5t}\left[\begin{pmatrix} 5 \\ -2 \end{pmatrix}\cos(4t) + \begin{pmatrix} 5 \\ 6 \end{pmatrix}\sin(4t)\right]$$

and the general solution:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$



5.2.21 Solve the system of equations:

$$x_1' = 5x_1 - 6x_3$$
$$x_2' = 2x_1 - x_2 - 2x_3$$
$$x_3' = 4x_1 - 2x_2 - 4x_3$$

$$\vec{x}' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{vmatrix} = (5-\lambda)[(-1-\lambda)(-4-\lambda)-4] - 6[-4-(-1-\lambda)4]$$

$$= (5-\lambda)(5\lambda+\lambda^2)+24+24(-1-\lambda)$$
$$= 25\lambda+5\lambda^2-5\lambda^2-\lambda^3-24\lambda$$
$$= -\lambda^3+\lambda = -\lambda(\lambda+1)(\lambda-1)$$

So, we have eigenvalues $\lambda = 0, -1, 1$.

With associated eigenvectors:

For $\lambda = 0$:

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$5a - 6c = 0$$
$$2a - b - 2c = 0$$
$$4a - 2b - 4c = 0$$

$a = 6$, $b = 2$, $c = 5$ work

$$\vec{v} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$$ is an associated eigenvector.

For $\lambda = -1$

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = - \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 6a \quad -6c &= 0 \\ 2a \quad -2c &= 0 \\ 4a - 2b - 3c &= 0 \end{aligned} \qquad \begin{aligned} a &= 2 \\ b &= 1 \\ c &= 2 \end{aligned} \quad \text{works}$$

$\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ is an associated eigenvector.

For $\lambda = 1$

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{aligned} 4a \quad -6c &= 0 \\ 2a - 2b - 2c &= 0 \\ 4a - 2b - 5c &= 0 \end{aligned} \qquad \begin{aligned} a &= 6 \\ b &= 2 \\ c &= 4 \end{aligned}$$

$\vec{v} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$ is an associated eigenvector.

So, we get the general solution:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} e^{t}}$$

5.2.39 Find the general solution of the system:

$$\vec{x}' = \begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \vec{x}$$

Finding the eigenvalues:

$$\begin{vmatrix} -2-\lambda & 0 & 0 & 9 \\ 4 & 2-\lambda & 0 & -10 \\ 0 & 0 & -1-\lambda & 8 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (-2-\lambda)[(2-\lambda)(-1-\lambda)(1-\lambda)]$$

$$= (-2-\lambda)(2-\lambda)(-1-\lambda)(1-\lambda)$$

So, we have eigenvalues $\lambda = 1, -1, 2, -2$ with associated eigenvectors:

$\lambda = 1$

$$\begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$a = 9$
$b = -8$ works, so $\vec{v} = \begin{pmatrix} 9 \\ -8 \\ 16 \\ 2 \end{pmatrix}$
$c = 16$
$d = 2$

$\lambda = -1$

$$\begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = - \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$a = 0$, $b = 0$, $c = 1$, $d = 0$ works so $\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 2$

$$\begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$a = 0$, $b = 1$, $c = 0$, $d = 0$ works so $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = -2$

$$\begin{pmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = -2 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$a = 1$, $b = 0$, $c = 0$, $d = 0$ works so $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

And we have the general solution:

$$\vec{x}(t) = c_1 \begin{pmatrix} 9 \\ -8 \\ 16 \\ 2 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-2t}$$

5.3.1 Find the two natural frequencies of the given system and describe its two natural modes of oscillation.

$m_1 = m_2 = 1 \qquad k_1 = 0,\ k_2 = 2,\ k_3 = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \quad \vec{x}'' = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 - 4 = \lambda^2 + 4\lambda$$

$$\lambda = 0, -4$$

with associated eigenvectors:

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} -2a + 2b = 0 \\ 2a - 2b = 0 \end{array} \qquad a = b \text{ so } \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

$\lambda = -4$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -4 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{matrix} 2a + 2b = 0 \\ 2a + 2b = 0 \end{matrix} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ works.}$$

So, the solution to our ODE is:

~~$\vec{x}(t) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t) + b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2t)$~~

$$\boxed{\vec{x}(t) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b_1 t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t) + b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2t)}$$

The natural ~~frequencise~~ frequencies are $\omega = 0$ and $\omega = 2$. In the $\omega = 0$ frequency it moves linearly in ~~and~~ one direction, while at $\omega = 2$ it oscillates in opposite directions with angular frequency $\omega = 2$

5.3.3 $m_1 = 1 \quad k_1 = 1$
$m_2 = 2 \quad k_2 = k_3 = 2$

So, we have the system of ODEs:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{x}'' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -4 \end{pmatrix} \vec{x} = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \vec{x}$$

we get eigenvalues:

$$\begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = (3+\lambda)(2+\lambda) - 2$$

$$= 6 + 5\lambda + \lambda^2 - 2 = \lambda^2 + 5\lambda + 4$$

$\lambda = -1, -4$ with eigenvectors

$$\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

~~$-4a + 2b = 0$~~ $\quad -2a + 2b = 0 \quad a = b$
~~$a - 3b = 0$~~ $\quad a - b = 0$

So,

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

$$\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -4\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} a + 2b = 0 \\ a + 2b = 0 \end{matrix} \qquad 2a = -b$$

So, $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ works.

So, the general solution is:

$$\boxed{\vec{x}(t) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t + b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t + a_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(2t) + b_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(2t)}$$

The two natural frequencies are $\omega = 1$ and $\omega = 2$. At $\omega = 2$ they oscillate in opposite directions with m_2 moving twice as far as m_1. At $\omega = 1$ they oscillate in the same direction with the same amplitude.

5.3.5. $m_1 = m_2 = 1 \qquad k_1 = 2, \; k_2 = 1, \; k_3 = 2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{x}'' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \vec{x}$$

this has eigenvalues:

$$\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (3+\lambda)^2 - 1 = \lambda^2 + 6\lambda + 8$$
$$= (\lambda+4)(\lambda+2)$$

So, we have eigenvalues $\lambda = -2, -4$.

The associated eigenvectors are:

$$\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -2 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{matrix} -a + b = 0 \\ a - b = 0 \end{matrix} \Rightarrow a = b \qquad \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ works.}$$

$$\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -4 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{matrix} a + b = 0 \\ a + b = 0 \end{matrix} \Rightarrow a = -b \qquad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ works.}$$

So, we get the general solution:

$$\vec{x}(t) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\sqrt{2}\,t) + b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\sqrt{2}\,t) + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t) + b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2t)$$

So, the natural frequencies are $\omega = \sqrt{2}$ and $\omega = 2$. At $\omega = \sqrt{2}$ the masses oscillate in the same direction with the same ~~frequency~~ amplitude. At $\omega = 2$ the masses oscillate in opposite directions but with the same amplitude.

5.3.9 The mass and spring system in problem 3 is set in motion from rest $(x_1'(0) = x_2'(0) = 0)$ in its equilibrium position $(x_1(0) = x_2(0) = 0)$ with the external forces $F_1(t) = 0$ and $F_2(t) = 120\cos(3t)$ acting on the masses m_1 and m_2, respectively. Find the resulting motion and describe it as a superposition of oscillations at three different frequencies

We get the system of ODEs:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}'' = \begin{pmatrix} -3 & 2 \\ 2 & -4 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 120 \end{pmatrix} \cos(3t)$$

$$\Rightarrow \vec{x}'' = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 60 \end{pmatrix} \cos(3t)$$

a particular solution to this ODE ~~#13~~:

$$\vec{x}_p = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cos(3t)$$

$$\vec{x}_p'' = -9\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\cos(3t)$$

$$= \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\cos(3t) + \begin{pmatrix} 0 \\ 60 \end{pmatrix}\cos(3t)$$

So,

$$-9c_1 = -3c_1 + 2c_2$$
$$-9c_2 = c_1 - 2c_2 + 60$$

$$\Rightarrow \quad 6c_1 + 2c_2 = 0$$
$$c_1 + 7c_2 + 60 = 0$$

$$\Rightarrow -\tfrac{1}{3}c_2 + 7c_2 + 60 = 0$$
$$\Rightarrow \tfrac{20}{3}c_2 = -60 \Rightarrow c_2 = -9 \qquad c_1 = 3$$

So,

$$\vec{x}(t) = a_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(t) + b_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin(t) + a_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}\cos(2t) + b_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}\sin(2t) + \begin{pmatrix} 3 \\ -9 \end{pmatrix}\cos(3t)$$

This is a superposition of oscillations with frequencies $\omega = 1$, $\omega = 2$, and $\omega = 3$.

Now, using our initial conditions

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 + 3 \\ a_1 - 2a_2 - 9 \end{pmatrix} \Rightarrow \sout{a_1 = 3, a_2 = -6} \quad a_1 = 1,\ a_2 = -4$$

and

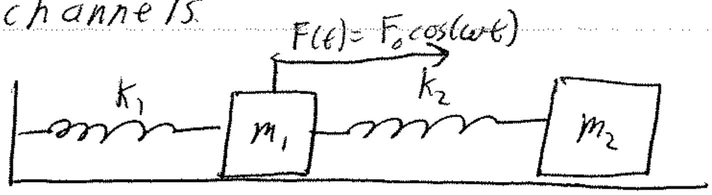
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 + 2b_2 \\ b_1 - 4b_2 \end{pmatrix} \Rightarrow b_1 = b_2 = 0.$$

So, our final solution is:

~~$\vec{x}(t) = 3\begin{pmatrix}1\\1\end{pmatrix}\cos(t) - 6\begin{pmatrix}1\\2\end{pmatrix}\cos(2t) + \begin{pmatrix}3\\9\end{pmatrix}\cos(3t)$~~

$$\boxed{\vec{x}(t) = \begin{pmatrix}1\\1\end{pmatrix}\cos(t) - 4\begin{pmatrix}1\\-2\end{pmatrix}\cos(2t) + \begin{pmatrix}3\\-9\end{pmatrix}\cos(3t)}$$

5.3.14 In the system below assume that $m_1 = 1$, $k_1 = 50$, $k_2 = 10$, and $F_0 = 5$ in mks units, and that $\omega = 10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest (!). Thus the effect of the second mass-and-spring pair will be to neutralize the effect of ~~this~~ the force on the first mass. This is an example of a <u>dynamic damper</u>. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.



We want to find the appropriate particular solution $\vec{x}_p$:

$$\vec{x}_p = \begin{pmatrix}c_1\\c_2\end{pmatrix}\cos(10t)$$

for the ODE:

$$\begin{pmatrix} 1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -60 & 10 \\ 10 & -10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix}\cos(10t)$$

$$\Rightarrow \vec{x}'' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}\begin{pmatrix} -60 & 10 \\ 10 & -10 \end{pmatrix}\vec{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix}\cos(10t)$$

~~$\Rightarrow \vec{x}'' = \Big(-60 + \frac{10}{m_2}$~~

$$\Rightarrow \vec{x}'' = \begin{pmatrix} -60 & 10 \\ \frac{10}{m_2} & -\frac{10}{m_2} \end{pmatrix}\vec{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix}\cos(10t)$$

So, plugging in $\vec{x}_p$:

$$-100\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\cos(10t) = \begin{pmatrix} -60 & 10 \\ \frac{10}{m_2} & \frac{-10}{m_2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\cos(10t) + \begin{pmatrix} 5 \\ 0 \end{pmatrix}\cos(10t)$$

So, we get the relations:

$$-100c_1 = -60c_1 + 10c_2 + 5$$
$$-100c_2 = \frac{10}{m_2}c_1 - \frac{10}{m_2}c_2$$

Now, we want $c_1 = 0$. If this is the case then $c_2 = -\frac{1}{2}$ and $\boxed{m_2 = .1}$

So, for $c_1 = 0$ (x_p has no movement of m_1) then $m_2 = .1$.